Symmetry Groups

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Abstract
This article presents mathematical structures that have algebraic properties similar to those found in numbers. In particular, we discuss the cube and its rotations.

We use numbers every day, whether it is at home, at work or even at the supermarket. In fact we use numbers so much that we seldom realize that there is much more to mathematics than just numbers and calculations. In this article I will present some examples of mathematical structures that use manipulations similar to those that we use for numbers.

Suppose we have two exact square pieces of paper placed on top of each other. The bottom square is not visible to us because it is completely covered by the top square. We can apply certain transformations to the top square such that we still cannot see the bottom square (without changing our viewing position). For example, if we rotate the top square 90 degrees clockwise (about its centre) then it will still completely cover the bottom square. However rotating the top square by just 45 degrees will make the bottom square visible.

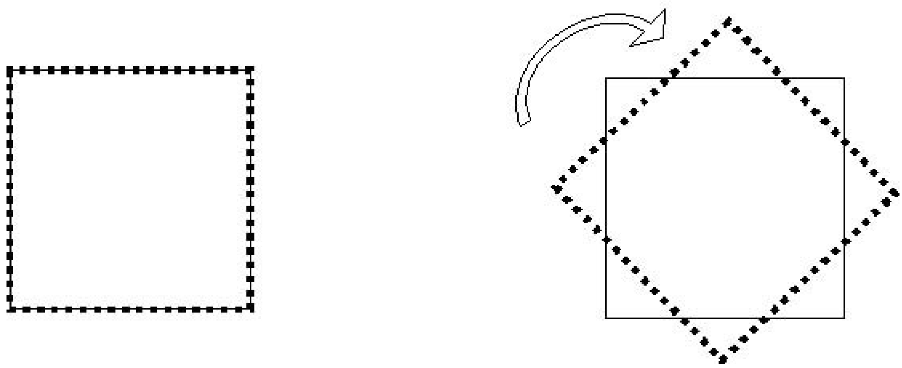


Fig 1: *Rotation by 90: Bottom square hidden* *Rotation by 45: Bottom square visible*

Whenever the bottom square remains hidden we say that the square has been preserved. In fact, we can preserve the square's position as long as we rotate it by a multiple of 90 degrees. This means we could rotate it by 90, 180, 270 or 360 degrees:

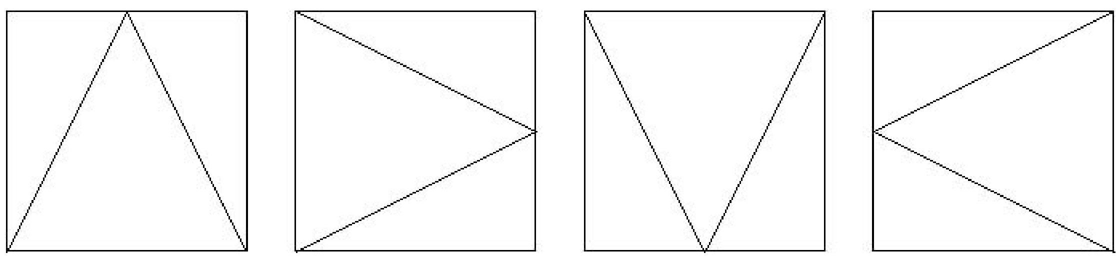
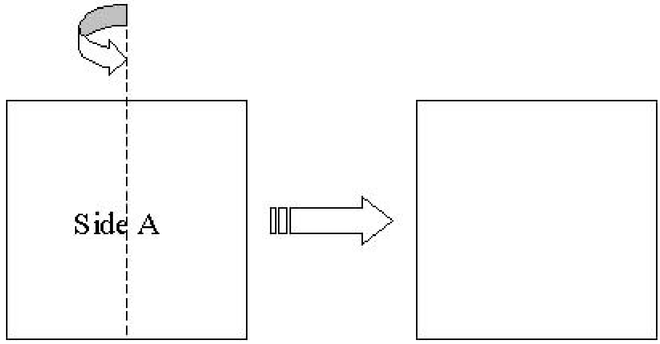


Fig.2: *Initial position (I)* *Rotated by 90* *Rotated by 180* *Rotated by 270*

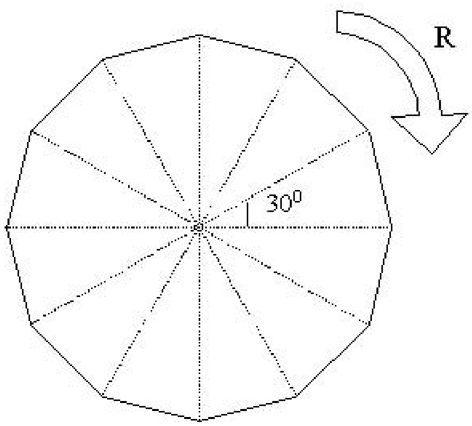
Is there anything else we can do to the top square piece to preserve its position? Not too surprisingly, we can also flip the square onto its opposite side (just like a pancake):



You may have already noticed that exactly 4 rotations (R) or 2 flips (F) bring us back to the initial position (I). Alternatively, this can be written as RRRR = FF = I. This method of writing allows us to express complex transformations in a short and concise manner. For example we can make R^n represent the act of rotating the square n times, and so RRRR becomes R^4.

Armed with this knowledge we are ready to begin our investigation and hopefully find some interesting results along the way. First of all, do R's and F's have anything in common which allows us to combine them? It turns out that a rotation followed by a flip is the same as a flip followed by 3 rotations. So now we have $RF = FR^3$ (similarly $FR = R^3F$). Do we need any other transformations to represent all of the square's positions? We can see that the total number of possible positions is 8, since there are 4 positions on side A and another 4 on side B. By rotations alone we can achieve 4 of these positions. Using flips only we can achieve 2 of these. So combining rotations and flips will give us 4 x 2 = 8 positions and that is all that we need. Finally we can show that any combination of transformations (whatever their order is) can be converted into one of the 8 distinct combinations: R, R^2, R^3, I, F, RF, R^2F, R^3F. For example, $FFR^3FR^2F = R^3FR^2F = R^3R^2FF = R^3R^2 = R$.

But why stop with a square? We can extend the argument to any n-sided regular polygon. This time around we rotate our piece by 360/n degrees for each rotation (R). The concept of a flip remains unchanged, except now we can flip along any of the diagonals. Here is an example with a 12-sided figure:



Here the rules are changed only slightly, with RF being equivalent to FR^{11}. In general, for an n-sided regular polygon $R^kF = FR^{n-k}$ and $FR^k = R^{n-k}F$. Also, there are 2n possible distinct positions. We can do something similar to any other 2-dimensional figure as long as it has some lines of symmetry. Below is a table for the symmetries found in the capital letters of the English alphabet:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
Flips	✓	✓	✓	✓	✓	×	×	✓	✓	×	✓	×	✓	×	✓	×	×	×	×	✓	✓	✓	✓	✓	✓	×
Rotations	×	×	×	×	×	×	×	✓	✓	×	×	×	×	✓	✓	×	×	×	✓	×	×	×	×	✓	×	✓

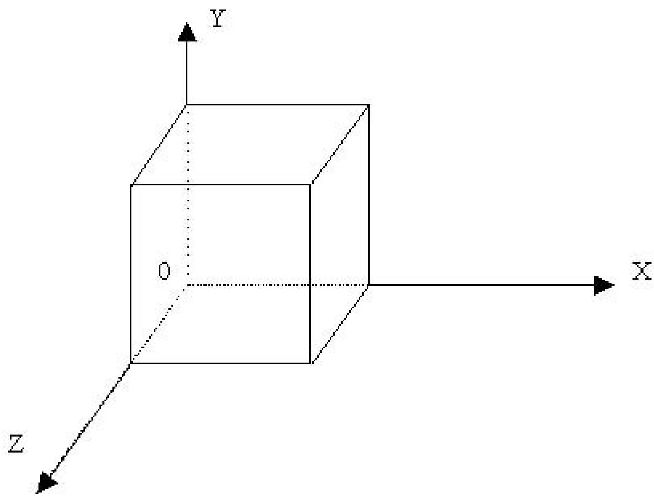
When dealing with numbers, we know that for any numbers A, B and C the following two rules always hold:

1. (A x B) x C = A x (B x C)
2. A x B = B x A

If we consider rule 1 (known as associativity) and replace A, B and C with some transformations then we will find that the rule still works with symmetry groups. However, this is not the case for rule 2 (known as commutativity) – we have already seen that RF = FR^{n-1} for an n-sided regular polygon and thus RF is not equivalent to FR.

Now that we have discussed what happens in 2 dimensions, it is time to expand our horizons and consider symmetry groups in 3 dimensions. In 3 dimensions, things become a lot more complicated, since we can now rotate objects along any of the 3 axes. Let's have a look at the simplest 3D object – the cube. If we glue the bottom face of the cube to a table then we can rotate the table 4 times before coming back to the original position. The same can be done to any other face of the cube. Since there are 6 faces, we have a total of 6 x 4 = 24 different positions of the cube. We now want to find some rotations of the cube, which will give us all of its 24 positions. However, this is not as easy as it sounds.

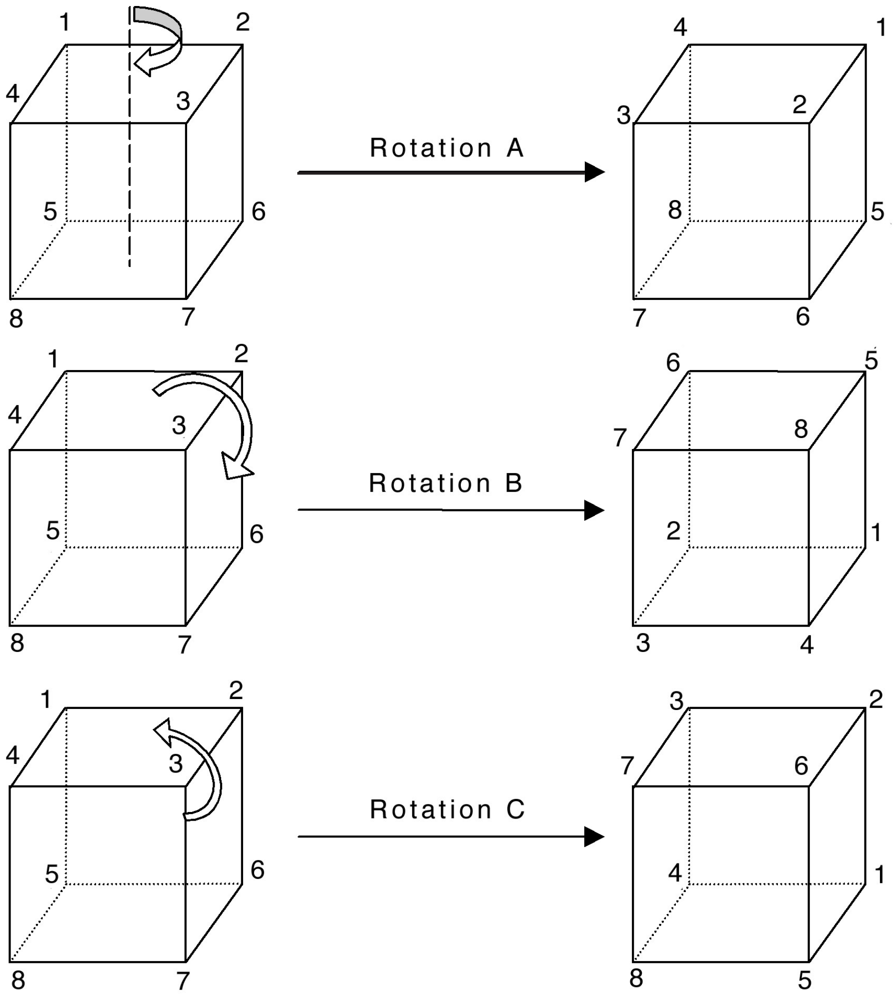
Using a naïve approach we could define 3 types of rotations. Each rotation will rotate the cube around one of the 3 axes:



Each type of rotation will produce exactly 4 positions of the cube. So combining all 3 types of rotations should give us 4 x 4 x 4 = 64 achievable positions. Since this number is greater than the total number of positions (24), we know that this naïve choice of rotations cannot uniquely represent every position of the cube.

Let's have a closer look at the number 24. It can be written as 4 x 2 x 3. Now we know that to achieve a unique representation of every position we need 3 types of rotations, where each type produces 4, 2 and 3 positions respectively.

Here I present an example of 3 such rotations. Rotation A involves rotating the cube 90 degrees clockwise around the line that joins the centres of face 1234 and face 5678. Rotation B involves rotating the cube 180 degrees around the line that joins the centres of face 3784 and face 2651. Rotation C involves rotating the cube 120 degrees into the page around the diagonal that joins vertices 2 and 8. We can see that $A^4 = B^2 = C^3 = I$.Although it is not totally obvious from the



diagrams, we can achieve any of the cube's 24 positions by applying a combination of A, B and C rotations only. Furthermore using this setup, every combination of A, B and C rotations can be uniquely represented as 3 or fewer A rotations followed by 1 or none B rotations followed by 2 or fewer C rotations. We will call this the standard form.

To convert any combination of rotations to standard form we need to apply the following rules:
1. $BA = A^3B$
2. $CA = A^3C^2$
3. $CB = A^2C$

For example $CBA = A^2CA = A^2A^3C^2 = AC^2$.

In conclusion, we have seen that symmetry groups use similar operations to those found in numbers. We have shown that some rules in numbers like associativity also hold in symmetry groups, while others like commutativity do not. We were able to define 3 types of cube rotations, such that there exists a unique way to represent all of the cube’s 24 positions.

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