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Stability of Linear Dynamical Systems with Memoryless Non-linearities†

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ABSTRACT

The Popov criterion for the stability of linear, time-invariant, finite-dimensional systems with a single non-linearity has been generalized by a numbers of authors through a relaxation of the single non-linearity and the finite-dimensional constraints in order to be applicable to a wider range of practical problems. This paper extends these results further by relaxing the time-invariant constraint. It is shown that a covariance condition when satisfied is a sufficient condition for the system output to be bounded and square integrable for zero input conditions.

§ 1. INTRODUCTION

THE problem of this paper is to establish a criterion for testing the stability of linear, time-varying, distributed parameter systems with memoryless non-linearities in a feedback loop (see the figure).

The Popov criterion for the stability of linear, time-invariant finite-dimensional systems containing a single non-linearity (Popov 1962) has been generalized by relaxing the finite dimensional constraint (Desoer 1965), the single non-linearity constraint (Anderson 1966 b, Moore and Anderson 1967 b) and both these constraints together (Anderson 1966 a). In the references already mentioned the stability test consists of checking that a particular function (or matrix function) is positive real. The case when a time-varying non-linearity is involved has been considered (Rekasius and Rowland 1965), and recently the more general case of time-varying, linear, finite-dimensional systems with memoryless non-linearities in the feedback loop has been considered (Moore and Anderson 1967 a). For this case the stability test consists in checking that a particular matrix function is a covariance.

In this paper the results given in Anderson (1966 a) are extended to the case when the distributed parameter linear sub-system is time varying. The stability criterion in this case is that a certain function be a covariance.

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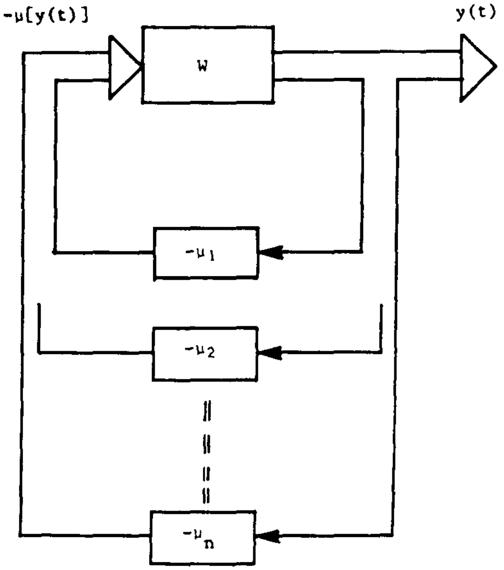
§ 2. Stability Criterion

The systems S under consideration have the form shown in the figure, where the linear, distributed parameter, time-varying sub-system W has an impulse response matrix $w(t, \tau)$ satisfying:

(S1) $\|w(t, \tau)\| < \alpha_1 \exp[-\alpha_2(t-\tau)]$, for all $t \geq \tau$ and some positive α_1 and α_2. It is also required that W maps piecewise continuous inputs into outputs whose derivatives are piecewise continuous.

It is further assumed that:

(S2) For any initial conditions, both the zero input response $y_0(t)$ of W and its derivatives $\dot{y}_0(t)$ are bounded (for $t \geq t_0$) and are square integrable on $[t_0, \infty)$ for all t_0.



System S.

The output $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))$ of system S is also the output of the sub-system W and the input to the non-linearities μ_i $(i = 1, 2, \ldots, n)$. The negative of the output from the non-linearities, $-\mu[y(t)]$ (written using abbreviated notation as $-\mu(t)$) is the input to the sub-system W. The non-linearities are memoryless and time-invariant, in the sense that μ_i depends explicitly only on y_i, rather than on y_i and t. The following condition is assumed:

(S3) $\mu'(t)y(t) \leq y'(t)Ky(t)$ for all t, or equivalently $\mu'(t)y(t) \geq \mu'(t)K^{-1}\mu(t)$ for all t where $\mu'(t) = (\mu_1(t), \mu_2(t), \ldots, \mu(t))$ and $K = \text{diag.}\{k_1, k_2, \ldots, k_n\}$ and is a constant positive definite matrix.

To develop a stability criterion for the above system, a hypothetical system Z is considered having an impulse response matrix:

$$z(t,\tau)=A(t)K^{-1}\delta(t-\tau)+A(t)w(t,\tau)+B(t)\frac{d}{dt}w(t,\tau), \tag{1}$$

where $\delta(t)$ is the Dirac delta function and

(Z1) $A(t)=\text{diag.}\{a_1(t),a_2(t),\ldots,a_n(t)\}$, and $B(t)=\text{diag.}\{b_1(t),b_2(t),\ldots,b_n(t)\}$ where for all i, $a_i(t)\geq 0$, $b_i(t)\geq 0$, $b_i(t)$ is differentiable with $\dot{b}_i(t)\leq 0$ and $a_i(t)$, $b_i(t)$ and $-\dot{b}_i(t)$ are bounded above for all t.

A matrix operator $R(t,\tau)$ is defined as:

$$R(t,\tau)=z(t,\tau)+z'(\tau,t). \tag{2}$$

The stability criterion to be established is that:

(Z2) $R(t,\tau)-2\eta I_n\delta(t-2\tau)$ is a covariance for some positive η and some A and B.

The condition (Z2) implies that:

$$\int_{t_0}^{t_1}\int_{t_0}^{t^+} x'(t)[R(t,\tau)-2\eta I_n\delta(t-\tau)]x(\tau)\,d\tau\,dt\geq 0, \tag{3}$$

which in turn implies that:

$$\int_{t_0}^{t_1}\int_{t_0}^{t^+} x'(t)[z(t,\tau)-\eta I_n\delta(t-\tau)]x(\tau)\,d\tau\,dt\geq 0, \tag{4}$$

where $x(\cdot)$ is an arbitrary vector function defined over $[t_0,t_1]$ with t_0 and t_1 arbitrary.

We now show for the system S (see the figure) satisfying (S1), (S2) and (S3) that if a modified system Z (see (1)) can be found such that (Z1) and (Z2) are satisfied, then the zero input response $y(\cdot)$ of S is bounded and square integrable on $[t_0,\infty)$ for any t_0.

Let the initial state of system W (at time t_0) be such that its zero input response is $y_0(t)$; then the response $y(t)$ of system S is given by:

$$y(t)=y_0(t)-\int_{t_0}^{t} w(t,\tau)\mu(\tau)\,d\tau. \tag{5}$$

We assume that $w(t,\tau)$ and μ regarded as a function of y are smooth enough to guarantee piecewise continuity of $\mu(\cdot)$. Then (S1) guarantees the existence of $\dot{y}(t)$ almost everywhere, where

$$\dot{y}(t)=\dot{y}_0(t)-\frac{d}{dt}\int_{t_0}^{t} w(t,\tau)\mu(\tau)\,d\tau. \tag{6}$$

Consider now the response of the modified system Z written using (1) in the form:

$$\int_{t_0}^{t^+}[z(t,\tau)-\eta I_n\delta(t-\tau)]\mu(\tau)\,d\tau=\int_{t_0}^{t^+}[A(t)K^{-1}-\eta I_n]\delta(t-\tau)\mu(\tau)\,d\tau$$
$$+A(t)\int_{t_0}^{t^+} w(t,\tau)\mu(\tau)\,d\tau+B(t)\frac{d}{dt}\int_{t_0}^{t^+} w(t)(t,\tau)\mu(\tau)\,d\tau. \tag{7}$$

Substituting (5) and (6) into (7) yields:

$$\int_{t_0}^{t^+} [z(t,\tau) - \eta I_n \delta(t-\tau)]\mu(\tau)\,d\tau = [A(t)K^{-1} - \eta I_n]\mu(t) + A(t)[y_0(t) - y(t)] + B(t)[\dot{y}_0(t) - \dot{y}(t)] \tag{8}$$

from which

$$\begin{aligned}\int_{t_0}^{t_1}\int_{t_0}^{t^+} \mu'(t)[z(t,\tau) - \eta I_n\delta(t-\tau)]\mu(\tau)\,d\tau\,dt &= \int_{t_0}^{t_1} \{\mu'(t)[A(t)K^{-1} - \eta I_n]\mu(t) - \mu'(t)A(t)y(t)\}\,dt \\ &\quad - \int_{t_0}^{t_1} \mu'(t)B(t)\dot{y}(t)\,dt + \int_{t_0}^{t_1} \mu'(t)[A(t)y_0(t) + B(t)\dot{y}_0(t)]\,dt.\end{aligned} \tag{9}$$

For convenience this equation is written $I_1 = I_2 + I_3 + I_4$ where each I_j corresponds to the appropriate integral in (9).

The application of (S3) and (Z1) to the integral I_2 yields:

$$I_2 \leq \int_{t_0}^{t_1} \{\mu'(t)[A(t)K^{-1} - \eta I_n]\mu(t) - \mu'(t)A(t)K^{-1}\mu(t)\}\,dt \tag{10}$$

or

$$I_2 \leq -\eta \int_{t_0}^{t_1} \mu'(t)\mu(t)\,dt. \tag{11}$$

We observe that if $\mu(\cdot)$ is not square integrable on $[t, \infty)$ then I_2 will diverge to $-\infty$ as t_1 increases.

In regard to I_3, since $B(t)$ is positive definite and bounded above and $\dot{B}(t)$ is non-positive definite (see (Z1)) the mean value theorem for integrals is applicable. Thus I_3 may be written as:

$$I_3 = -\int_{t_0}^{T} \mu'(t)B(t_0)\dot{y}(t)\,dt, \tag{12}$$

for some T satisfying $t_0 \leq T \leq t_1$. Introducing a change of variable yields:

$$I_3 = \int_{y(T)}^{y(t_0)} \mu'(y)B(t_0)\,dy, \tag{13}$$

This may be re-written as:

$$I_3 = \int_{0}^{y(t_0)} \mu'(y)B(t_0)\,dy - \int_{0}^{y(T)} \mu'(y)B(t_0)\,dy. \tag{14}$$

As t_1 increases the second term in the equation may diverge. If it does, I_3 will diverge to $-\infty$ by (S3).

The divergence properties of I_4 may be examined using the Cauchy–Schwarz inequality, i.e.

$$|I_1| \leq \left\{ \int_{t_0}^{t_1} \mu'(t)\mu(t)\,dt \right\}^{1/2} \left\{ \int_{t_0}^{t_1} [A(t)y_0(t) + B(t)\dot{y}_0(t)]' \times [A(t)y_0(t) + B(t)\dot{y}_0(t)]\,dt \right\}^{1/2}. \quad (15)$$

This may be simplified to:

$$|I_4| \leq \left[\int_{t_0}^{t_1} \mu'(t)\mu(t)\,dt \right]^{1/2} M, \quad (16)$$

where M is a constant independent of t_1 since $A(\cdot)$ and $B(\cdot)$ are bounded (see (Z1)), and $y_0(\cdot)$ and $\dot{y}_0(\cdot)$ are square integrable on $[t_0, \infty)$, (see (S2)). Certainly I_4 may diverge to $+\infty$ as t_1 increases if $\mu(\cdot)$ is not square integrable; but we observe that I_2 diverges to $-\infty$ at a faster rate. Thus $I_2 + I_3 + I_4$ will diverge to $-\infty$ as t_1 increases if $\mu(\cdot)$ is not square integrable.

We conclude that (S2), (S3), (Z1) and (Z2) imply that $\mu(\cdot)$ *is square integrable on* $[t_0, \infty)$ for any t_0 in order that (9), (11), (13) and (16) be satisfied simultaneously; for if $\mu(\cdot)$ is not square integrable on $[t_0, \infty)$, then $(I_2 + I_3 + I_4)$ diverges to $-\infty$ as t_1 increases and this contradicts (Z2).

The above result enables us to establish that $y(\cdot)$ is bounded and square integrable on $[t_0, \infty)$ for all t_0.

Consider the integral in (5). An application of the Cauchy–Schwarz inequality gives:

$$\left| \int_{t_0}^{t} w(t,\tau)\mu(\tau)\,d\tau \right| \leq \left[\int_{t_0}^{t} w'(t,\tau)w(t,\tau)\,d\tau \right]^{1/2} \left[\int_{t_0}^{t} \mu'(\tau)\mu(\tau)\,d\tau \right]^{1/2} \quad (17)$$

and it is seen that the square integrability of $\mu(\cdot)$ and condition (S1) give an upper bound for the integral in (17) and thus in (5). This result together with the boundedness of $y_0(\cdot)$ (see (S2)) implies directly from (5) that $y(\cdot)$ is bounded for all t_0.

The assumption (S1) implies that a square integrable input $\mu(\cdot)$ to system W results in a square integrable output $y(\cdot)$ on $[t_0, \infty)$. The reasoning is as follows:

$$\|y(t)\| \leq \left\| \int_{t_0}^{t} w(t,\tau)\mu(\tau)\,d\tau \right\| \quad (18)$$

$$\leq \int_{t_0}^{t} \|w(t,\tau)\| \quad \|\mu(\tau)\|\,d\tau. \quad (19)$$

Using (S1):

$$\|y(t)\| \leq \int_{t_0}^{t} \alpha_1 \exp[-\alpha_2(t-\tau)]\|\mu(\tau)\|\,d\tau. \quad (20)$$

Since $\mu(\cdot)$ is square integrable so is $\|\mu(\cdot)\|$. Tichmarsh (1962, see theorem 65) then yields that the the integral on the right of (20), regarded as a function of t, is square integrable on $[t_0, \infty)$ and thus $y(\cdot)$ is square integrable on $[t_0, \infty)$ for all t_0.

The above results thus have established the stability criterion.

Stability Criterion. Systems having the structure as illustrated in the figure where

(i) the sub-system W is linear (possibly distributed and time-varying) and satisfies (S1) and (S2); and

(ii) the memoryless sector non-linearities are such that (S3) is satisfied

have a bounded and square integrable zero input response on $[t_0, \infty)$ for any initial conditions and initial time t_0, provided the following condition is satisfied:

(Z2) $R(t, \tau) - \eta I_n \delta(t-\tau)$ is a covariance from some positive η and some $A(t)$ and $B(t)$ satisfying (Z1) where $R(t, \tau)$ is given from (1) and (2).

We observe that for the case $B(t)=0$ a time-varying non-linearity satisfying (S3) does not affect the development of the above stability criterion since I_3 (see (12)) for this case is zero and the other terms are not affected. Moreover, the restriction given in the second part of (S1) is not required. (This property is perhaps of greater interest in the case when the sub-system W is time invariant).

§ 3. Concluding Remarks

For the case when the sub-system W is finite dimensional it may be represented by the state-space equations:

$$\dot{x} = Fx + Gu,$$

$$y = H'x.$$

In this case, sufficient conditions for (S1) and (S2) to be satisfied are that F, G, H are bounded, $[F, G]$ is uniformly completely controllable, $[F, H']$ is uniformly completely observable, and the transition matrix $\Phi(t, \tau)$ of W is exponentially bounded, i.e.

$$\|\Phi(t, \tau)\| \leq \alpha_3 \exp[-\alpha_4(t-\tau)],$$

for some positive α_3 and α_4. The stability criteria in this case correspond to those given for such systems in Moore and Anderson (1967 a).

For the case when the sub-system W is time invariant, the covariance condition reduces to a positive real condition and condition (S1) is simplified to requiring that the poles of the elements of the transfer function matrix $W(s)$ of the sub-system W be in the left half plane $\operatorname{Re} s < 0$ and that $W(\infty) = 0$. The stability criterion in this case becomes that given in Anderson (1966 a).

For the case when both finite-dimensional and time-invariant constraints are imposed, the criterion corresponds to the multiple non-linearity Popov criterion (Anderson 1966 b, Moore and Anderson 1967 b) which in turn corresponds to the well known Popov criterion for the single loop case (Popov 1962).

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