Complex Plane and Parameter Plane Linear System Design Methods

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Summary

Constraint equations and inequalities relating the adjustable parameters of a system are considered for a multi-parameter design. Constraints determined from the characteristic equation root specifications, dominant root parameter sensitivity specifications and error constant specifications are incorporated into the design analytically, while relative stability and dominancy constraints are considered using graphical methods on the parameter and complex planes.

1. Introduction

The problem considered in this paper is the efficient calculation and presentation of information to provide a basis for the selection of the adjustable parameters in a control system design.

The Siljak parameter plane method[1, 2] gives relative stability information for linear systems as a function of two system parameters. Using a digital computer, this information is readily calculated even for high order multi-loop systems. Other performance characteristics such as the system error constant,[3] bandwidth[4] and dominant root sensitivity[5] may also be plotted on the parameter plane, thus enabling the designer to select values for the two adjustable parameters which give a design compromise. The transfer function pole/zero locations may be determined at any point in the parameter plane by mapping a grid of contours covering the complex plane into the parameter plane and using interpolation. Having this information is useful as the well established correlations between pole/zero patterns, frequency characteristics and time domain performance may be used. However, the calculations in question and their interpretation for high order systems are inefficient and the possibility of improving the design with further parameter changes or structural changes is not indicated.

In this paper, the parameter plane concept is extended to facilitate an efficient multi-parameter design. In selecting the adjustable parameters of a system, constraints determined from the characteristic equation root specifications,[1] error constant specifications[3] and dominant root parameter sensitivity specifications are incorporated into the design analytically, while relative stability and dominancy constraints are considered using graphical methods on the parameter and complex planes.

2. Basic Considerations

2.1 *Constraint Equations*

2.1.1 *Characteristic Equations*

The characteristic equation considered in the design of continuous linear systems may be written as

$$f(s) = \sum_{k=0}^{n} a_k s^k = 0, \qquad a_k \neq 0 \tag{1}$$

where s is the complex variable ($s = \sigma + j\omega$) and the coefficients, a_k, are functions of the r system adjustable parameters, $q_1, q_2, \ldots q_r$, that is, $a_k = a_k(q_1, q_2, \ldots q_r)$.

Using the functions $X_k(\sigma, \omega)$ and $Y_k(\sigma, \omega)$ defined from the equation

$$s^k = X_k + j\,Y_k \tag{2}$$

equation 1 may be conveniently separated into two equations by equating the real part, R, and the imaginary part, I, of f(s) to zero, that is,

$$R = \sum_{k=0}^{n} a_k X_k = 0 \tag{3}$$

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1. Siljak, D. D., " Analysis and Synthesis of Feedback Control Systems in the Parameter Plane—Part I, Linear Continuous Systems ", *I.E.E.E. Trans. on Applic. and Ind.*, Vol. 83, No. 75, November 1964, p. 449.
2. Siljak, D. D., " Generalization of the Parameter Plane Method ", *Trans. I.E.E.E.*, Vol. AC-11, No. 1, January 1966, p. 63.
3. Moore, J. B., " Steady-State Response in the Parameter Plane ", *Teorijska Automatika*, Vol. 1, No. 2, August 1965, p. 55.
4. Hollister, F. H. and Thaler, G. J., " Loaded and Null Adjusted Symmetrical Parallel-Tee Network ", Vol. 21, Proceedings of National Electronics Conference, I.E.E.E., Chicago, October 1965, p. 753.
5. Siljak, D. D. and Burzio, A., " Minimization of Sensitivity Constraints in Linear Control Systems ", *Trans. I.E.E.E.*, Vol. AC-11, No. 3, July 1966, p. 567.

$$I = \sum_{k=0}^{n} a_k Y_k = 0$$

For the case when s is real, that is, $\omega = 0$, equation 3 reduces to

$$R = \sum_{k=0}^{n} a_k \sigma^k = 0 \tag{4}$$

The functions X_k and Y_k may be calculated using recurrence relationships,

$$\begin{aligned} X_{k+2} - 2\sigma X_{k+1} + (\sigma^2 + \omega^2) X_k &= 0 \\ Y_{k+2} - 2\sigma Y_{k+1} + (\sigma^2 + \omega^2) Y_k &= 0 \end{aligned} \tag{5}$$

where $X_0 \equiv 1$, $X_1 \equiv \sigma$, $Y_0 \equiv 0$ and $Y_1 \equiv \omega$.

It is seen that equations 3 and 4 are in fact constraint equations relating the system parameters for the case when characteristic equation complex roots (that is, $s = \sigma \pm j\omega$) and real roots (that is, $s = \omega$) are specified respectively. Equation 3 is more convenient to use that equation 1 because in equation 3 the expressions are given in the real domain.

2.1.2 *Root Sensitivity Equations*

If the sensitivity of a characteristic equation complex root is specified as well as the root itself, four constraint equations result.

Consider that the sensitivity of the real and imaginary parts of a complex root of equation 1 are defined as

$$S_\sigma = \frac{\Delta\sigma/\sigma}{\Delta q_i/q_i} \quad \text{and} \quad S_\omega = \frac{\Delta\omega/\omega}{\Delta q_i/q_i} \tag{6}$$

so that when σ, ω, S_σ, S_ω, q_i and Δq_i are specified, the constraint expressions are given by equation 3 together with the following equations :

$$\begin{aligned} \sum_{k=0}^{n} a_k' X_k (\sigma + \Delta\sigma, \omega + \Delta\omega) &= 0 \\ \sum_{k=0}^{n} a_k' Y_k (\sigma + \Delta\sigma, \omega + \Delta\omega) &= 0 \end{aligned} \tag{7}$$

where $a_k' = a_k(q_1, q_2, \ldots, q_{i-1}, q_i + \Delta q_i, q_{i+1}, \ldots q)$ and

$$\begin{aligned} \Delta\sigma &= \sigma S_\sigma \Delta q_i / q_i \\ \Delta\omega &= \omega S_\omega \Delta q_i / q_i \end{aligned}$$

If the sensitivity of a characteristic equation real root is specified as well as the root itself, two constraint equations result, that is, equation 4 and the following :

$$\sum_{k=0}^{n} a_k' (\sigma + \Delta\sigma)^k = 0 \tag{8}$$

When more than the one parameter is considered as variable, that is, $q_i (i = 1, 2, \ldots r)$ the sensitivity functions (equation 6) may be generalised to, for example,

$$S_\sigma = \frac{\Delta\sigma/\sigma}{\sum_{i=1}^{r} \Delta q_i/q_i}, \qquad S_\omega = \frac{\Delta\omega/\omega}{\sum_{i=1}^{r} \Delta q_i/q_i}.$$

and a_k' written as

$$a_k' = a_k (q_1 + \Delta q_1, q_2 + \Delta q_2, \ldots q_r + \Delta q_r)$$

For the case when the classical small parameter sensitivities are specified, equations 7 and 8 may still be used by selecting the parameter variation (for example, Δq_1) to be arbitrarily small.

2.1.3 *Error Constant Equations*

The constraint equation relating the system parameters for the case when an error constant is specified is readily derived from the appropriate error constant definition, that is,

$$\begin{aligned} K_p &= \lim_{s \to 0} G(s) \\ K_v &= \lim_{s \to 0} s\,G(s) \\ K_a &= \lim_{s \to 0} s^2 G(s) \end{aligned} \tag{9}$$

where K_p, K_v and K_a are the position error constant, velocity error constant and acceleration error constant, respectively, and G(s) is the open loop system transfer function.

2.2 *Constraint Inequalities*

2.2.1 *Relative Stability Constraints on the Parameter Plane*

For the case when the coefficients, a_k, of equation 1 can be expressed as linear functions of two parameters, q_1 and q_2, and their product term, $q_1 q_2$, that is,

$$a_k = b_k q_1 + c_k q_2 + d_k q_1 q_2 + e_k \tag{10}$$

as often occurs in the characteristic equation of linear control systems with two adjustable parameters, then equation 3 may be rewritten as follows,

$$q_1 \sum_{k=0}^{n} b_k X_k + q_2 \sum_{k=0}^{n} c_k X_k + q_1 q_2 \sum_{k=0}^{n} d_k X_k + \sum_{k=0}^{n} e_k X_k = 0 \tag{11a}$$

$$q_1 \sum_{k=0}^{n} b_k Y_k + q_2 \sum_{k=0}^{n} c_k Y_k + q_1 q_2 \sum_{k=0}^{n} d_k Y_k + \sum_{k=0}^{n} e_k Y_k = 0 \tag{11b}$$

For a specified σ and ω, the X_k and Y_k functions may be readily calculated using equation 5 and thus equations 11(a) and 11(b) may be solved for the two parameters, q_1 and q_2. Repeated calculations enable contours in the complex s-plane to be plotted into the $q_1 q_2$ parameter plane. Shading the parameter plane contours according to the sign of the Jacobian of equations 11 (a) and 11(b), enables the regions in the parameter plane to be interpreted in terms of the number of roots within the corresponding regions of the complex plane.[6] For the interpretation to be complete, prior information is required concerning the location of the roots for one point in the parameter plane.

If contours covering the entire complex plane are mapped into the parameter plane, the roots of the characteristic equation at any point in the parameter plane will be given in evidence using interpolation. The value of such results for a control system design is apparent but the inefficiency of such calculations is also evident.

The important contributions of this method is that

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[6] Moore, J. B., "Circle Criteria in the Parameter Plane", *Proc. I.E.E.*, Vol. 115, No. 4, April 1968, p. 577.

regions of a specified relative stability are given in evidence in a parameter plane from just a few calculations. If more information is required regarding the location of the roots, the results are best presented in the complex plane.

2.2.2 *Relative Stability and Dominancy Constraints on the Complex Plane*

The two parameter control system problem may be considered in the complex plane. The solution of equations 11(a) and 11(b) together with the application of a method for factoring polynomials gives the parameters q_1 and q_2 which achieve a specified root together with the remaining roots. The repeated application of this procedure enables a root locus to be plotted having a specified branch (for example, a constant damping ratio contour) and for each point in the locus two parameters are calculated. The plotting of just one, two or three such root loci is sufficient to indicate whether or not the two adjustable parameters can be adjusted to satisfy the relative stability and dominancy requirements.

The advantage of using the above procedures is that relative stability and dominancy constraints may be conveniently considered simultaneously. Further, by plotting the loci for other values of the fixed parameters, the effects of a third parameter are considered conveniently on the same complex plane.

The shading of the root loci according to the sign of the product of two Jacobians indicates the movement of the loci for an incremental change of the specified loci. This may be useful in interpreting the complex plane root loci for a system design.

It is to be noted that the zeros of a polynomial may be found in a straightforward convergent procedure which has rapid convergence in the region of a zero using the X_k and Y_k functions.[7] Thus, the solution of the parameter plane equations followed by the solution of the reduced polynomial is both an efficient and a useful calculation.

3. Design

The simultaneous solution of any available independent constraint equations may be used to determine suitable values for the adjustable parameters. For the general case when the equations are non-linear, if more than two real solutions occur, the interpretation of the results may be difficult. For the case when the equations are linear or contain only one product term, straightforward general computer programs may be used giving, at the most, two real solutions. Thus, in order to apply the general computer programs only two, three, four or maybe five parameters are considered.

The following steps may be used,

1. Calculate the coefficients of the adjustable parameters in the constraint equations (the X_k and Y_k functions will be used when the constraint equations are of the form of equations 3 or 7 and these are readily calculated using equation 5).
2. Solve the simultaneous constraint equations for the adjustable parameters and calculate any Jacobian that may be required (note : only real solutions have significance).
3. Construct the characteristic equation of the system from the results, remove any known factors of this equation and then calculate the remaining factors.
4. Calculate any other relevant information such as relative stability, dominancy, system zeros, sensitivity characteristics, error constant, bandwidth, stochastic properties, time domain criteria or the system response itself.
5. Repeat the process for alternate specifications or other fixed parameters or other system structures until a satisfactory design compromise is achieved.

It is at this point in the calculations that the designer may employ graphical methods using the complex and parameter planes and contribute to the design procedure from his experience or his ability to interpret preceding results in terms of what further investigations are required.

Examples of the application of the above steps will now be given.

Example 1

(*i*) *Parameter Plane*

For the multi-loop system of fig. 1, it is required to investigate the effects of variations of the parameters q_1 and q_2 on the system relative stability for the case $q_3 = 1.25$. The system characteristic equation is given as

$$(19.2q_3)s^5 + (19.2 + 74.5q_3)s^4 + (74.5 + 75.7q_3 + 1440q_1)s^3 + \\ + (75.7 + 23.6q_3 + 792q_1 + 1228q_2q_3)s^2 + \\ + (23.6 + 461q_3 + 36q_1 + 1228q_2)s + 460 = 0 \qquad (12)$$

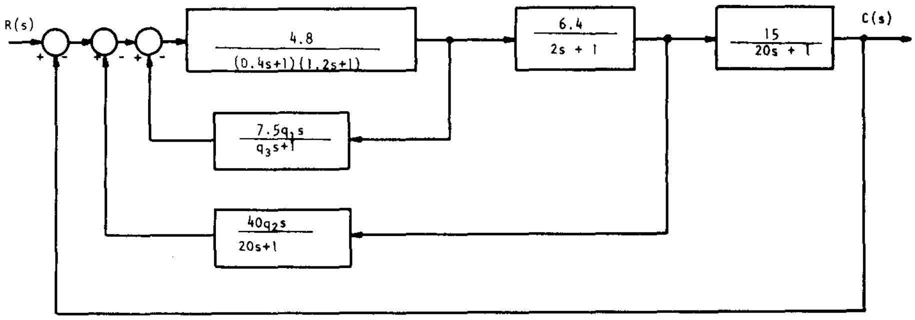

Figure 1.—*System block diagram.*

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[7] Moore, J. B., " A Convergent Algorithm for Solving Polynomial Equations ", *J. of A.C.M.*, Vol. 14, No. 2, April 1967, p. 311.

Fig. 2 gives one set of contours which cover the parameter plane plotted using as constraint equations, equations 11(a) and 11(b). The regions of various relative stabilities are thus readily determined. In order to give further information regarding the root locations, further contours must be plotted and the roots at any point determined using interpolation.

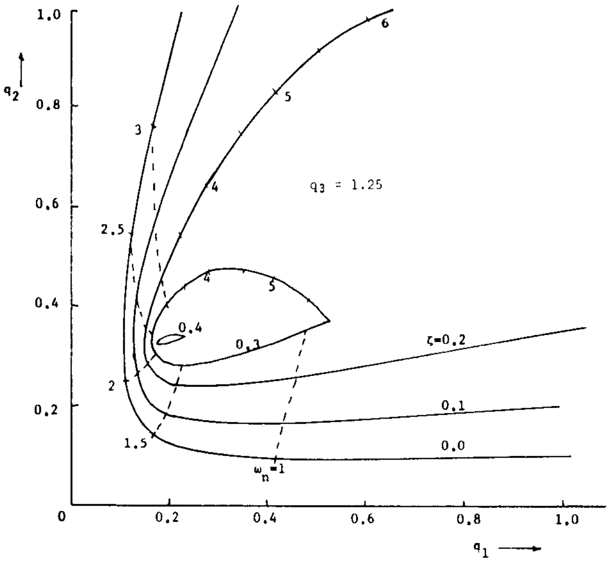

Figure 2.—Parameter plane diagram.

Fig. 3 gives another set of contours which cover the parameter plane. For high order systems the plotting of remaining contours may require a considerable search procedure and it may be required to plot many planes of the form of figs. 2 and 3 in order to avoid intersection of many contours. The plotting of information regarding root configurations is more efficient on the complex plane.

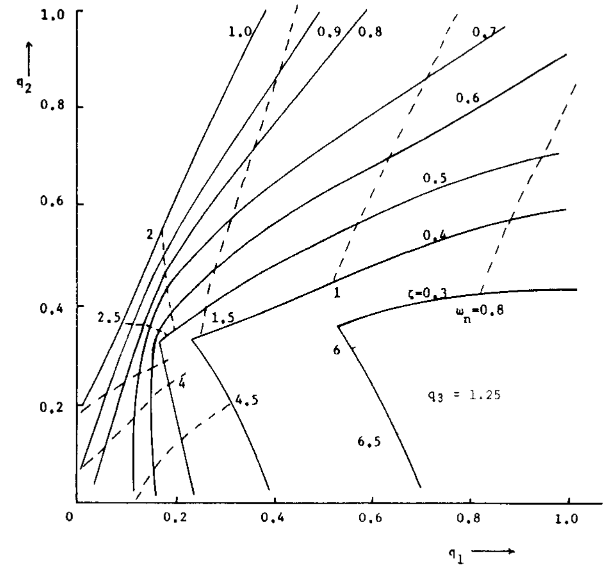

Figure 3.—Parameter plane diagram.

(ii) Complex Plane

For the example above using steps 1, 2 and 3 of the preceding section, root loci are plotted on the complex plane having the relative stability contour $\zeta = -\sigma/(\sigma^2 + \omega^2)^{\frac{1}{2}} = 0.3$ as a specified branch with natural frequency $\omega_n = (\sigma^2 + \omega^2)^{\frac{1}{2}}$ as a parameter. The loci are plotted for the case when $q_3 = 0.5$ and when $q_3 = 1.25$, (see fig. 4). At any point in the loci, the two parameters q_1 and q_2 are given from the $\zeta = 0.3$ loci in fig. 5. The transient responses corresponding to the points marked 1-5 on the loci are given in fig. 6 for the case $q_3 = 1.25$. It is seen that responses 3 and 4 give good compromises between rise time, overshoot and settling time. If the response was not acceptable, repeating the calculations for either different fixed parameters or for different specified branches enables a system design to be carried out in a systematic manner. A further example of the complex plane approach is considered in reference 8.

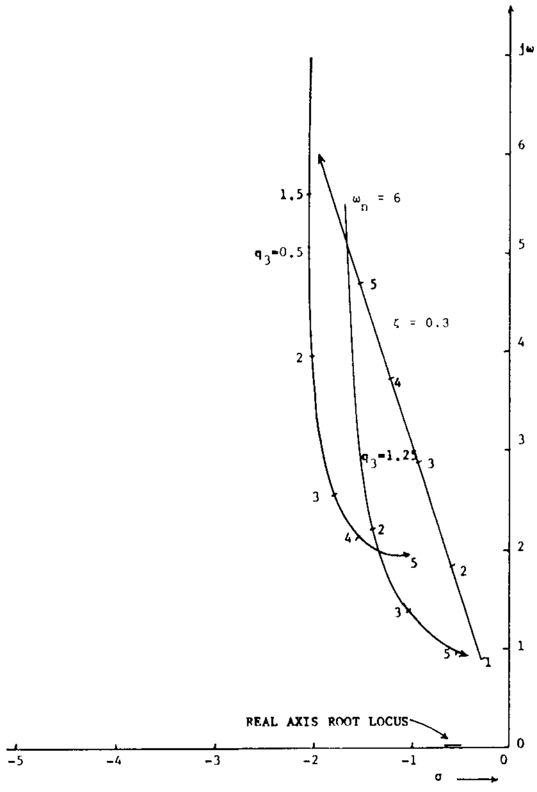

Figure 4.—Complex plane loci.

Example 2

This example illustrates the application of steps 1, 2 and 3 for the case when more than two constraint equations are considered.

(i) *Specified Dominant Root Sensitivity*

Consider the system of fig. 7 in which the feedback

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8. Moore, J. B. and Dorf, R. C., "The Design of an Attitude Control System for a Space Vehicle", Proceedings of the NEC Conference, October 1966.

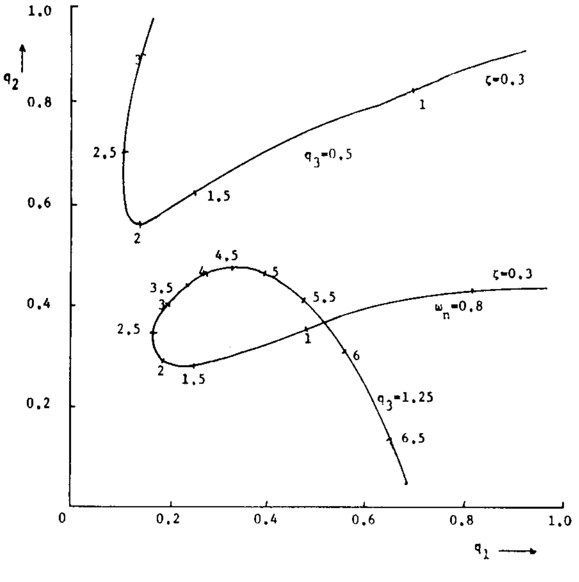

Figure 5.—Parameter plane $\zeta = 0.3$ contours.

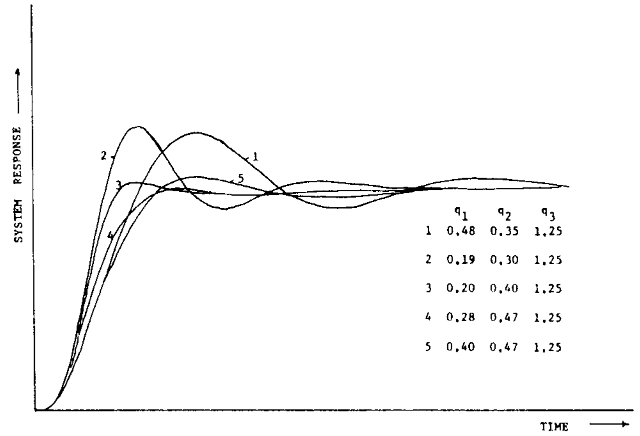

Figure 6.—System transient response.

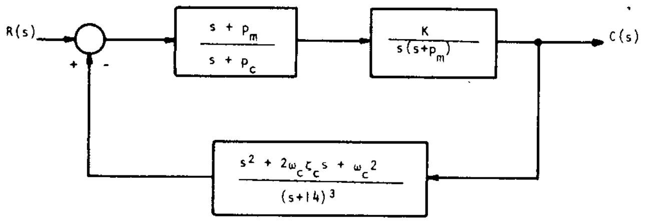

Figure 7.—System block diagram.

compensation poles are at $s = -14$. The zeros of the feedback compensator, ζ_c, ω_c, the pole of the series compensator, p_c, and the system gain, K, are to be determined such that the dominant roots vary from $A(-5 \pm j6)$ to $B(-2 \pm j2)$ as the gain, K, decreases by a factor of ten, that is, from K_A to K_B.

The system characteristic equation is

$$s^5 + (42 + p_c)s^4 + (588 + 42p_c)s^3 + (2744 + 588p_c + K)s^2 + \\ + (2744p_c + 2\zeta_c\omega_c K)s + K\omega_c^2 = 0 \qquad (13)$$

The coefficients, a_k, may be expressed as linear functions of the parameters q_1, q_2, q_1q_2, q_3 and q_4 where $q_1 = K_A$, $q_2 = \omega_c^2$, $q_3 = p_c$ and $q_4 = K_A\zeta_c\omega_c$. The four equations (5 and 11) are now solved for the four parameters q_1, q_2, q_3 and q_4 where $\sigma = -5$, $\omega = 6$, $\sigma' = -2$, $\omega' = 2$ and $a_k' = a_k(0.1q_1, q_2, q_3, 0.1q_4)$. From the results, the following values were calculated. $K_A = 4357$, $K_B = 435.7$, $\omega_c = 7.45$, $\zeta_c = 0.754$ and $p_c = 4.37$. The other roots of the characteristic equation corresponding to the case when the dominant roots are at $A(-5 \pm j6)$ were calculated as $-3.6 \pm j\,11.12$. The results of the approximate root locus method by Horowitz[9] are given for comparison.

$$K_A = 4600,\ K_B = 460,\ \omega_c = 7.5,\ \zeta_c = 0.707,\ p_c = 4.7.$$

(ii) *Specified Dominant Root*

For the system of the above problem (that is, fig. 7 and equation 13), if two pairs of roots are specified the four parameters once again may be calculated, using two pairs of equations given from equation 3. If required, a parameter plane may be plotted of the form of fig. 2 or a complex plane may be plotted of the form of fig. 4 for which a specified root exists and for which, at each point in the plane, four parameters are given in evidence.

(iii) *Specified Error Constant*

Either of the error constant expressions (equation 9) may be solved simultaneously with the parameter plane equations for three adjustable parameters quite readily. An example is given in reference 3.

4. Conclusions

This paper has presented an approach to the design of multi-parameter systems using analytical and graphical techniques.

The introduction of constraint equations to be considered in conjunction with the parameter plane equations has been investigated but shown to be of limited usefulness except when the constraint equation is the error constant equation.

Considerations of graphical procedures based on straightforward computer calculations gave the following conclusions. Parameter plane plots are useful when relative stability is specified and when the effects of parameters on system characteristics are to be investigated. On the other hand, complex plane plots are useful, particularly for a multi-parameter design when the pole/zero locations must satisfy both relative stability and dominancy requirements.

The significant characteristics of the procedures presented are that the multi-parameter problems are considered systematically, at each step in the design a more optimum solution is given and possible further investigations are indicated. This conclusion is further illustrated in the design of a sixth order multi-loop system in reference 7.

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[9] Horowitz, I. M., " Synthesis of Feedback Systems ", Academic Press, New York (1963), Chapters 6 and 9.