Extensions of quadratic minimisation theory: the discrete-time case

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Abstract

A covariance condition is shown to be a necessary condition for the existence of a finite control law associated with discrete-time quadratic loss-minimisation problems. Sufficient conditions (including the covariance condition) are then determined which extend the range of optimal-control problems, for which a well defined control law may be calculated. The results may also be applied to solving problems in filtering and stability theory.

1 Introduction

The problem considered in this paper is the determination of a set of necessary and sufficient conditions for the existence of a well defined control law associated with discrete-time quadratic loss-minisation problems.

The discrete-time systems considered are linear and finite-dimensional, and may be represented by the state-space equation

$$x(k+1) = F(k)x(k) + G(k)u(k) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where x is an $n$vector (the state), u is an $m$vector (the input), F and G are matrices of appropriate dimension, and the indexing set $\{k\}$ is the set of integers. The quadratic performance index considered is

$$V\{x(k_0), u, k_1, k_0\} = \sum_{k=k_0}^{k_1-1} \{u'(k)R(k)u(k) + 2x'(k)S(k)u(k) + x'(k)Q(k)x(k)\} + x'(k_1)Ax(k_1) \quad (2)$$

where the matrices $F(k)$, $G(k)$, $R(k)$, $S(k)$, $Q(k)$ are finite-valued, and, without loss of generality, $R(k) = R'(k)$, $Q(k) = Q'(k)$ for all $k \in [k_0, k_1 - 1]$ with $k_0 \in \{k\}$, and $k_1 \in \{k\}$ and $A = A'$.

When the above conditions are satisfied, $A > 0$, $R > 0$, $S = 0$ and $Q \geqslant 0$ [here $X > 0 (X \geqslant 0)$ means that X is positive (semi)definite], it is straightforward to apply Bellman's principle of optimality to determine a unique, finite-valued control law u, which minimises the performance index given by eqn. 2 (see Reference 1). The main calculation involved is the recursive solution of a matrix Riccati difference equation. There are often times (including important cases of practical value) when the above sufficient conditions, for the minimisation process to be well defined, are not satisfied and yet the various calculations yield the desired results. If the inverse operation in the calculations is replaced by the pseudoinverse operation, the range of optimal-control problems for which a solution can be calculated is extended.[2] When the pseudoinverse operation is used, the control law calculated gives the minimum-energy control which will minimise the quadratic performance index; any control which simply minimises the performance index is not then unique.

What are the necessary and sufficient conditions for the application of the optimal-control equations, including the case when a pseudoinverse operation is used, to yield finite solutions? This question has been answered, for a corresponding continuous-time optimal-control problem, in previous papers;[3,4] some insight into aspects of the discrete-time problem has been gained from this work, but most of the material in the papers is unrelated. Applications of the continuous-time results, to solve the time-varying spectral-factorisation problem,[5] has enabled a number of linear-filtering and estimation problems to be solved.[6–8] Also, some results in stability theory have been developed by direct use of the theory given in Reference 9. Corresponding application of the discrete-time results of this paper, to the discrete-time, time-varying spectral-factorisation problem and its applications, is considered in a companion paper.[10]

In Section 2, discrete-time optimal-control results are reviewed, and some extensions are given when a pseudoinverse operation is used instead of the regular inverse operation.

In Section 3, it is shown that a covariance condition is necessary for the minimisation process to be well defined. Sufficient conditions, based on this result, are also developed. Appendix 6 gives an alternative interpretation for the solution of the Riccati equation when pseudoinverse operations are involved.

2 Preliminary considerations

It is first assumed that the minimum performance index $\bar{V}\{x(k), k_1, k\} = \min_{u} V\{x(k), u, k_1, k\}$ has the form

$$\bar{V}\{x(k), k_1, k\} = x'(k)\Pi(k-1, k_1)x(k) \quad . \quad . \quad . \quad (3)$$

where $\Pi(k-1, k_1)$ is finite for $k \in [k_0, k_1 - 1]$. Implicit in this assumption is the assumption that the optimal-control law u is finite. Application of Bellman's principle of optimality to eqn. 3 gives the following results:

$$\begin{aligned}\bar{V}\{x(k), k_1, k\} &= \min_{u(k)} [u'(k)R(k)u(k) + 2x'(k)S(k)u(k) \\ &\quad + x'(k)Q(k)x(k) + \bar{V}\{F(k)x(k) \\ &\quad + G(k)u(k), k_1, k+1\}] \quad . \quad . \quad . \quad . \quad (4) \\ &= \min_{u(k)} \{u'(k)B(k)u(k) + 2x'(k)C(k)u(k) \\ &\quad + x'(k)D(k)x(k)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)\end{aligned}$$

$$\text{where } \left.\begin{aligned} B &= R + G'\Pi G \\ C &= S + F'\Pi G \\ D &= Q + F'\Pi F \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

When $B(k)$ is *positive definite*, with $k \in [k_0, k_1 - 1]$, the optimal-control law $\bar{u}$, i.e. that control which minimises $u'Bu + 2x'Cu + x'Dx$, is given immediately from eqn. 5 as

$$u = -B^{-1}C'x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and $\Pi(k-1, k_1)$ is given recursively from the Riccati equation

$$\left.\begin{aligned} \Pi(k-1, k_1) &= D(k) - C(k)\{B(k)\}^{-1}C'(k) \\ \Pi(k_1 - 1, k_1) &= A \end{aligned}\right\} \quad . \quad . \quad (8)$$

Normally, when $A > 0$, $R > 0$, $S = 0$ and $Q \geqslant 0$, then $\bar{V} \geqslant 0$ (see eqn. 2), and thus $\Pi(\cdot, k_1) \geqslant 0$ (see eqn. 3). Again, since $R > 0$ and $\Pi(\cdot, k_1) \geqslant 0$, eqn. 6 gives $B > 0$. Thus the conditions just given are sufficient conditions for the optimal-control problem to be solved using eqns. 3 and 6–8.

There are many cases when these sufficient conditions are not satisfied and yet the various calculations yield the desired results. Replacing the inverse operation in eqns. 7 and 9 permits an even wider range of problems to be solved. The

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theory behind the use of the pseudoinverse operation is now reviewed.

The performance index given by eqn. 5 may be rewritten for all $k \in [k_0, k_1]$ as

$$\bar{V}\{x(k), k_1, k\} = \min_{u} \{(u + B\dagger C'x)'B(u + B\dagger C'x) + x'(D - CB\dagger C')x + 2x'C(I - B\dagger B)u\} \quad . \quad . \quad (9)$$

The index k has been omitted for convenience, and † has been used to denote the pseudoinverse operation. [Important properties are that $B\dagger = B\dagger BB\dagger$, $B = BB\dagger B$ and $(B\dagger)\dagger = B$.] Implicit in eqn. 9 is the assumption that $\bar{V}\{x(k), k_1, k+1\}$, so that $\bar{V}\{x(l), k_1, l\}$ for $l \in [k+1, k_1]$ exists as a finite-valued minimum performance index. The control u may be expressed as $u = u_1 + u_2$, where $u_1 \in \mathscr{R}(B)$ (the range space of B) and $u_2 \in \mathscr{N}(B)$ (the null space of B). Since B is symmetric $\mathscr{R}(B)$ and $\mathscr{N}(B)$ are orthogonal, and thus

$$u'u = u_1'u_1 + u_2'u_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Rewriting eqn. 9 is terms of u_1 and u_2 gives[2]

$$\bar{V}\{x(k), k_1, k\} = \min_{u} \{(u_1 + B\dagger C'x)'B(u_1 + B\dagger C'x) + x'(D - CB\dagger C')x + 2x'Cu_2\} \quad . \quad . \quad (11)$$

Results are now established, using the above equation as a starting point, which lead directly to a constructive procedure and to existence results, which may be used in solving the optimal-control problem when B may be singular.

Necessary and sufficient conditions for $\bar{V}\{x(k), k_1, k\}$ to be bounded in terms of $x(k)$ are that $B \geqslant 0$ and $\mathscr{N}(B) \subseteq \mathscr{N}(C)$. The conditions are necessary since, if they are not satisfied, either a u_1 or a u_2 may be chosen so that $\bar{V}$ is arbitrarily negative. That the conditions are sufficient may be seen as follows. Since $u_2 \in \mathscr{N}(B) \subset \mathscr{N}(C)$, $2x'Cu_2 = 0$; and, since $B \geqslant 0$, $u_1 = -B\dagger C'x$ minimises the first term on the right-hand side of eqn. 11, which may be rewritten simply as

$$\bar{V} = x'(D - CB\dagger C')x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

This is bounded below in terms of $x(k)$. The result is thus established.

Eqn. 12 may be written, for $k = k_0$, as

$$\bar{V}\{x(k_0), k_1, k_0\} = x'(k_0)\Pi(k_0 - 1, k_1)x(k_0) \quad . \quad (13)$$

where $\Pi(k-1, k_1)$ is redefined as

$$\left.\begin{aligned} \Pi(k-1, k_1) &= D(k) - C(k)\{B(k)\}\dagger C'(k) \\ \Pi(k_1 - 1, k_1) &= A \end{aligned}\right\} \quad . \quad (14)$$

There are certainly many u which minimise the performance index when the above conditions are satisfied and $B(k)$ is singular. However, it is evident from eqn. 10 that the control u, for $u_2 = 0$, is the minimal-energy control. Thus the optimal-control law u is usually chosen as u_1, i.e.

$$u = -B\dagger C'x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The result is now established that $\mathscr{N}(B) \subset \mathscr{N}(C)$ if, and only if, $C(I - B\dagger B) = 0$. Suppose $x \in \mathscr{N}(B)$ and $C(I - B\dagger B) = 0$; then $0 = C(I - B\dagger B)x = Cx$, and thus $x \in \mathscr{N}(C)$. Conversely, suppose $x \in \mathscr{N}(B)$ implies that $x \in \mathscr{N}(C)$. If y is an arbitrary vector decomposed as $y = x + z$, where $x \in \mathscr{N}(B) \subset \mathscr{N}(C)$ and $z \in R(B)$, $C(I - B\dagger B)y$ reduces to zero.

The various results given above are now summarised as a lemma.

Lemma 1: Consider the system of eqn. 1 and the performance index (eqn. 2), with the conditions, given immediately after eqn. 2, satisfied. Consider, also, the Riccati equation, eqn. 14 (see also eqns. 6), and the control law $\bar{u}$ given by eqn. 15. Necessary and sufficient conditions for $\bar{u}$ to be the minimal-energy control which minimises eqn. 2 are that $B > 0$, *or* $B \geqslant 0$ and $C(I - B\dagger B) = 0$ at each iteration, or, equivalently, that a minimum performance index is known to exist and be bounded below. The minimum index is given by eqn. 13 (see also eqns. 6 and 14).

From the above lemma, it is evident that the *necessary and sufficient conditions* for the optimal-control solution to exist and be finite are *those which establish a lower bound* (independent of the control) *on the performance index*. These latter conditions are determined in Section 3.

3 Existence results

Consider the linear difference equation

$$\left.\begin{aligned} P(k-1) &= F'(k)P(k)F(k) + Q(k) \\ P(k_1 - 1) &= A \end{aligned}\right\} \quad . \quad . \quad . \quad (16)$$

Using eqn. 16, it is possible to expand $\sum_{k=k_0}^{k_1-1} x'(k)Q(k)x(k)$ as follows:

$$\sum_{k=k_0}^{k_1-1} x'(k)Q(k)x(k) = \sum_{k=k_0}^{k_1-1} x'(k)\{P(k-1) - F'(k)P(k)F(k)\}x(k)$$

Application of eqn. 1 gives

$$\sum_{k=k_0}^{k_1-1} x'(k)Q(k)x(k) = \sum_{k=k_0}^{k_1-1} [x'(k)P(k-1)x(k) - \{x(k+1) - G(k)u(k)\}'P(k)\{x(k+1) - G(k)u(k)\}]$$

Some cancellation of terms and a further application of eqn. 1 yields

$$\sum_{k=k_0}^{k_1-1} x'(k)Q(k)x(k) = x'(k_0)P(k_0 - 1)x(k_0) - x'(k_1)P(k_1 - 1)x(k_1) + 2\sum_{k=k_0}^{k_1-1} [x'(k)\{F'(k)P(k)G(k)\}u(k) + \tfrac{1}{2}u'(k)\{G'(k)P(k)G(k)\}u(k)]$$

This means that if H and J are defined as

$$\left.\begin{aligned} H &= S + F'PG \\ J &= \tfrac{1}{2}(R + G'PG) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

the index given by eqn. 2 may be written as

$$V\{x(k_0), u, k_1, k_0\} = x'(k_0)P(k_0 - 1)x(k_0) + 2\sum_{k=k_0}^{k_1-1} \{u'(k)J(k) + x'(k)H(k)\}u(k) \quad . \quad (18)$$

If y, an $n$vector, is defined by

$$y(k) = H'(k)x(k) + J(k)u(k) \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

then, for $x(k_0) = 0$, eqn. 18 reduces to

$$V(0, u, k_1, k_0) = \sum_{k=k_0}^{k_1-1} \{y'(k)u(k) + u'(k)y(k)\} \quad . \quad . \quad . \quad . \quad (20)$$

The output y of the system, having the state of eqns. 1 and 19 {for $x(k_0) = 0$}, may be written in terms of the system-impulse response $w(k, l)$ as follows:

$$y(k) = \sum_{l=k_0}^{k} w(k, l)u(l) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where

$$w(k, l) = J(k)\delta(k - l) + H'(k)\Phi(k, l+1)G(l)\mathbf{1}(k - l) \quad (22)$$

The terms $\mathbf{\Phi}$, $\boldsymbol{\delta}$ and $\mathbf{1}$ are the state-transition matrix of eqn. 1, the unit-delta function and the unit-step function, respectively. They are defined by the equations

$$\begin{aligned} \Phi(k+1, l) &= F(k)\Phi(k, l) \\ \Phi(l, l) &= I \\ \mathbf{1}(k - l) &= \begin{cases} 1 \text{ for } k > l \\ 0 \text{ for } k \leqslant l \end{cases} \\ \delta(k - l) &= \begin{cases} 1 \text{ for } k = l \\ 0 \text{ for } k \neq l \end{cases} \end{aligned} \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Since $w(k, l) = 0$ when $l > k$, a rearrangement of eqn. 20 using eqns. 21 and 22 may be made as follows:

$$V(0, u, k_1, k_0) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} u'(k)R_y(k, l)u(l) \quad . \quad (24)$$

where $\boldsymbol{R}_y(k, l) = \boldsymbol{w}(k, l) + \boldsymbol{w}'(l, k)$ . . . . . . (25)

and $\boldsymbol{u}(k)$ is zero outside the range $[k_0, k_1 - 1]$. Direct application of the covariance property, i.e. that $\boldsymbol{R}_y(k, l)$ is a covariance on $[k_a, k_b]$ if, and only if,

$$\sum_{k_a}^{k_b} \sum_{k_a}^{k_b} \boldsymbol{u}'(k)\boldsymbol{R}_y(k, l)\boldsymbol{u}(l) \geqslant 0 \quad . \; . \; . \; . \; . \; . \quad (26)$$

for an arbitrary choice of $\boldsymbol{u}$, gives lemma 2.

Lemma 2: Consider the system of eqn. 1 and the performance index (eqn. 2) with the conditions, given immediately after eqn. 2, satisfied. A necessary and sufficient condition for the performance index $V(0, \boldsymbol{u}, k_1, k_0)$ (see eqn. 24) to be nonnegative for all $\boldsymbol{u}$, is for $\boldsymbol{R}_y(k, l)$ to be a covariance on $[k_0, k_1]$, where

$$\begin{aligned}\boldsymbol{R}_y(k, l) = {} & 2\boldsymbol{J}(k)\delta(k - l) + \boldsymbol{H}'(k)\boldsymbol{\Phi}(k, l + 1)\boldsymbol{G}(l)\mathbf{1}(k - l) \\ & + \boldsymbol{G}'(k)\boldsymbol{\Phi}'(l, k + 1)\boldsymbol{H}'(l)\mathbf{1}(l - k) \quad . \; . \; . \; . \quad (27)\end{aligned}$$

(see also eqns. 16, 17 and 23).

Results for the more general case, when $\boldsymbol{x}(k_0) \neq 0$, are now considered.

Lemma 3: For the system of eqn. 1 with the performance index (eqn. 2) and the conditions, given immediately after eqn. 2, satisfied, sufficient conditions for a lower bound on $V\{\boldsymbol{x}(k_0), \boldsymbol{u}, k_1, k_0\}$ to exist for all $\boldsymbol{u}$ and $\boldsymbol{x}(k_0)$ [dependent only on $\boldsymbol{x}(k_0)$ and k_0] are that

$\{\boldsymbol{R}_y(k, l) - \eta \boldsymbol{I}\delta(k - l)\}$ is a covariance on $[k_0, k_1]$ for some positive constant η.

Proof: Application of lemma 2 gives the result that, for all $\boldsymbol{u}$, if there is a $\boldsymbol{u}_c$ and a $\boldsymbol{k}_c \in (k)$ with $k_c < k_0$ so that, for some selection of $\boldsymbol{F}$ and $\boldsymbol{G}$ made on $[k_c, k_0]$, the system goes from the zero state at k_c to the state $\boldsymbol{x}(k_0)$ at k_0, and if selections of $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{S}$ on $[k_c, k_0]$ can be made so that $\boldsymbol{R}_y(k, l)$ is a covariance on $[k_c, k_1]$, then

$$V\{\boldsymbol{x}(k_0), \boldsymbol{u}, k_1, k_0\} \geqslant \boldsymbol{K}\{\boldsymbol{x}(k_0), k_0\} > -\infty$$

where $\boldsymbol{K}$ is $-V(0, \boldsymbol{u}_c, k_0, k_c)$ and depends only on k_0 and $\boldsymbol{x}(k_0)$, since $\boldsymbol{u}_c$ depends on $\boldsymbol{x}(k_0)$.

It is now shown that the condition given at the end of lemma 3 implies these above assumptions. This will complete the proof of the lemma. Choose k_c, $\boldsymbol{F}(k)$ and $\boldsymbol{G}(k)$ for $k \in [k_c, k_0]$ so that $[\boldsymbol{F}, \boldsymbol{G}]$ is completely reachable at k_0, and let $\boldsymbol{H}(k) = 0$ and $\boldsymbol{R}(k) = \rho \boldsymbol{I}$ for $k \in [k_c, k_0]$, where ρ is a positive constant. Then, for an arbitrary choice of $\boldsymbol{u}$, the following expansion holds:

$$\begin{aligned}\sum_{k=k_c}^{k_1-1} \sum_{l=k_c}^{k_1-1} \boldsymbol{u}'(k)\boldsymbol{R}_y(k, l)\boldsymbol{u}(l) = {} & \rho \sum_{k=k_c}^{k_0} \boldsymbol{u}'(k_0 - 1)\boldsymbol{u}(k_0 - 1) \\ & + \sum_{k=k_0}^{k_1-1} \sum_{k=k_0}^{k_1-1} \boldsymbol{u}'(k)\boldsymbol{R}_y(k, l)\boldsymbol{u}(l) \\ & + 2\sum_{k=k_0}^{k_1-1} \boldsymbol{u}'(k)\boldsymbol{H}'(k)\boldsymbol{\Phi}(k, k_0) \\ & \quad \sum_{l=k_c}^{k_0} \boldsymbol{\Phi}(k_0, l)\boldsymbol{G}(l)\boldsymbol{u}(l)\end{aligned}$$

Using similar calculations to those in the corresponding continuous-time problem,[3] applying the Cauchy inequality and using the condition given at the end of lemma 3, yields

$$\sum_{k=k_0-1}^{k_1-1} \sum_{k=k_0-1}^{k_1-1} \boldsymbol{u}'(k)\boldsymbol{R}_y(k, l)\boldsymbol{u}(l) \geqslant \rho z_1^2 + \eta z_2^2 - \gamma z_1 z_2 \quad . \quad (28)$$

$$\begin{aligned}\text{where } z_1^2 &= \boldsymbol{u}'(k_0 - 1)\boldsymbol{u}(k_0 - 1) \\ z_2^2 &= \sum_{k=k_0}^{k_1-1} \boldsymbol{u}'(k)\boldsymbol{u}(k) \\ \gamma &= \text{constant.}\end{aligned}$$

Hence, there exists a choice of ρ such that the right-hand side of eqn. 28 is nonnegative definite {i.e. so that $\boldsymbol{R}_y(k, l)$ is a covariance on $[k_0 - 1, k_1]$} irrespective of z_1 or z_2, and thus of $\boldsymbol{u}$. If η were zero, this could not be done. This completes the proof of the lemma.

It is now evident from the results of Section 2, together with the above theorem, that the necessary and sufficient conditions for the optimal-control law to be well defined are established.

Before summarising the above results, a further result is now proved which clarifies the understanding of the problem for certain cases.

Lemma 4: For the system of eqn. 1, with the performance index (eqn. 2) and the conditions given immediately after eqn. 2 satisfied, and if $\boldsymbol{R}(k)$ is nonsingular for some $k \in [k_0, k_1 - 1]$, then, with $\boldsymbol{\Pi}$ calculated as in eqn. 14 (see also eqn. 6), the performance index $V\{\boldsymbol{x}(k), \boldsymbol{u}, k_1, k\}$ cannot be bounded below for some $\boldsymbol{x}(k)$ and all $\boldsymbol{u}$ unless $\boldsymbol{B}(k)$, given by eqn. 6, is positive definite. Thus, if $\boldsymbol{R}(k)$ is nonsingular for $k \in [k_0, k_1 - 1]$, then a necessary and sufficient condition for the optimal-control solution to exist and be finite is that $\boldsymbol{B}(k) > 0$ for $k \in [k_0, k_1 - 1]$.

Proof: When $\boldsymbol{R}$ is nonsingular, without loss of generality we may assume that $\boldsymbol{S} = 0$. (If $\boldsymbol{R}$ is nonsingular, the control $\boldsymbol{u}$ may be replaced by $\boldsymbol{u} + \boldsymbol{R}^{-1}\boldsymbol{S}'\boldsymbol{x}$ and $\boldsymbol{Q}$ replaced by $\boldsymbol{Q} - \boldsymbol{S}\boldsymbol{R}^{-1}\boldsymbol{S}'$). When, for some $\boldsymbol{x}(k)$, $V\{\boldsymbol{x}(k), \boldsymbol{u}, k_1, k\}$ is bounded below for all $\boldsymbol{u}$, then from lemma 1, $\mathcal{N}\{\boldsymbol{B}(k)\} \subset \mathcal{N}\{\boldsymbol{C}(k)\}$. From eqn. 6 and the fact that $\boldsymbol{F}$ is nonsingular, it is evident that this, in turn, implies that $\mathcal{N}\{\boldsymbol{B}(k)\} \subset \mathcal{N}\{\boldsymbol{C}(k)\} \subset \mathcal{N}\{\boldsymbol{R}(k\}$. This cannot be satisfied with $\boldsymbol{B}(k)$ singular and $\boldsymbol{R}(k)$ nonsingular, and so it is concluded that, if $\boldsymbol{R}(k)$ is nonsingular and $\boldsymbol{B}(k)$ is singular, then $V\{\boldsymbol{x}(k), \boldsymbol{u}, k_1, k\}$ cannot be bounded below for all $\boldsymbol{x}(k)$ and $\boldsymbol{u}$. This result, together with lemma 1, implies that, if $\boldsymbol{R}(k)$ is nonsingular for $k \in [k_0, k_1 - 1]$, a necessary and sufficient condition for the optimal-control solution to exist and be finite is that $\boldsymbol{B}(k) > 0$ for $k \in [k_0, k_1 - 1]$.

The results of lemmas 1–4 may now be summarised as a theorem.

Theorem 1: For the system of eqn. 1 with the performance index given by eqn. 2, the necessary and sufficient conditions for a control law $\bar{\boldsymbol{u}}$, which minimises eqn. 2, to be finite are that *either* $\boldsymbol{B} > 0$ *or* $\boldsymbol{B} \geqslant 0$ and $\boldsymbol{C}(\boldsymbol{I} - \boldsymbol{B}^\dagger \boldsymbol{B}) = 0$ for all $k \in [k_0, k_1 - 1]$, where $\boldsymbol{B}$ and $\boldsymbol{C}$ are calculated from eqns. 6 and 14. [When $\boldsymbol{R}(k)$ is nonsingular, these conditions are reduced and simply require that $\boldsymbol{B} > 0$.] Alternatively, a necessary condition for $\bar{\boldsymbol{u}} = -\boldsymbol{B}^\dagger \boldsymbol{C}'\boldsymbol{x}$ to be the minimising control law is that $\boldsymbol{R}_y(k, l)$ be a covariance (see eqns. 27, 16 and 17); sufficient conditions are that those conditions, given at the end of lemma 3, should be satisfied. When the above sufficient conditions are satisfied, the minimal-energy control $\bar{\boldsymbol{u}}$ is given by $\bar{\boldsymbol{u}} = \boldsymbol{B}^\dagger \boldsymbol{C}'\boldsymbol{x}$ and the minimum index is given by eqn. 13.

4 Conclusions

The use of Bellman's principle of optimality to solve the linear, discrete, quadratic-minimisation problem is well known. This paper focused attention on necessary and sufficient conditions which ensure that the various procedures work even for singular cases. It was observed that tests may be made at each step in the calculations to guarantee that the minimisation process is meaningful. Three matrices ($\boldsymbol{B}$, $\boldsymbol{B}^\dagger$ and $\boldsymbol{C}$, given in the calculations, were involved. It was required that either $\boldsymbol{B} > 0$ or that $\boldsymbol{B} \geqslant 0$, and that $\boldsymbol{C}(\boldsymbol{I} - \boldsymbol{B}^\dagger \boldsymbol{B}) = 0$. (If $\boldsymbol{R} > 0$, it would require simply that $\boldsymbol{B} > 0$.)

The paper has also shown that a covariance condition involving the various matrices $\boldsymbol{F}$, $\boldsymbol{G}$, $\boldsymbol{R}$, $\boldsymbol{S}$, $\boldsymbol{Q}$ and $\boldsymbol{A}$ is a necessary condition for the optimal-control law to be finite. Sufficient conditions based on this result are also given. Appendix 6 shows that these conditions also ensure that a particular Riccati difference equation has a well defined solution, even when one of the terms involves the inverse of a singular matrix. The dual of this result, obtained by a time-reversal process, is the key existence result used in a discrete spectral-factorisation procedure.[10] The results have a direct application to the solution of the time-varying version of the discrete, multiple-input, Popov stability criterion.[11]

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6 Appendix

This Appendix develops a result which gives an alternative interpretation for the solution of the Riccati equation (eqn. 19) when $\boldsymbol{B}$ is singular. A preliminary result is as follows.

Corollary to lemma 3: Consider the system of eqn. 1 with performance index given by

$$V\{\boldsymbol{x}(\hat{k}), \boldsymbol{u}, k_1, \hat{k}, \epsilon\} = V\{\boldsymbol{x}(\hat{k}), \boldsymbol{u}, k_1, \hat{k}\} + \epsilon \boldsymbol{u}'(\hat{k})\boldsymbol{u}(\hat{k}) \quad . \quad . \quad . \quad . \quad (29)$$

with $\hat{k} \in [k_0, k_1]$ and ϵ a positive constant. The minimum performance index (over all $\boldsymbol{u}$), written $\bar{V}\{\boldsymbol{x}(\hat{k}), k_1, \hat{k}, \epsilon\}$, is bounded above for all $\epsilon \geqslant 0$. The sufficient conditions for this index to be bounded below by a constant, dependent only on $\boldsymbol{x}(\hat{k})$ and $\hat{k}$, are those of lemma 3 with k_0 replaced by $\hat{k}$.

Proof: Since $\boldsymbol{u} \equiv 0$ is a legitimate control, an upper bound on $V\{\boldsymbol{x}(\hat{k}), \boldsymbol{u}, k_1, \hat{k}, \epsilon\}$ is $\boldsymbol{x}'(\hat{k})\boldsymbol{P}(\hat{k} - 1)\boldsymbol{x}(\hat{k})$ (see eqns. 29 and 18). To establish a lower bound, if the sufficient conditions stated in lemma 3 are satisfied, they will still be satisfied with $\boldsymbol{R}(\hat{k})$ replaced by $\boldsymbol{R}(\hat{k}) + \epsilon \boldsymbol{I}$ for $\epsilon > 0$. Applying lemma 3 shows that, under the conditions of lemma 3, a lower bound exists for the performance index (eqn. 29) dependent only on $\boldsymbol{x}(\hat{k})$ and $\hat{k}$.

The expression for the minimum performance index $\bar{V}\{\boldsymbol{x}(\hat{k}), k_1, \hat{k}, \epsilon\}$ is given by

$$\bar{V}\{\boldsymbol{x}(\hat{k}), \hat{k}_1, \hat{k}, \epsilon\} = \boldsymbol{x}'(\hat{k})\boldsymbol{\Pi}(\hat{k} - 1, k_1, \epsilon)\boldsymbol{x}(\hat{k}) \quad . \quad (30)$$

$$\left.\begin{aligned} \text{where } \boldsymbol{\Pi}(k - 1, k_1, \epsilon) &= \boldsymbol{D}(k) - \boldsymbol{C}(k)\{\boldsymbol{B}(k) \\ &\quad + \epsilon \boldsymbol{I}\boldsymbol{\delta}(k - \hat{k})\}^{-1}\boldsymbol{C}(k)' \\ \boldsymbol{\Pi}(k_1 - 1, k_1, \epsilon) &= \boldsymbol{A} \end{aligned}\right\} \quad . \quad . \quad (31)$$

Lemma 5: Under the conditions given at the end of lemma 3,

$$\lim_{\varepsilon \to 0} \boldsymbol{\Pi}(\hat{k} - 1, k_1, \epsilon) = \boldsymbol{\Pi}(\hat{k} - 1, k_1) \quad . \quad . \quad (32)$$

where $\boldsymbol{\Pi}(\epsilon)$ is given from eqn. 31 and $\boldsymbol{\Pi}$ is given from eqn. 14. This result is interesting, as certainly $\lim_{\varepsilon \to 0} (\boldsymbol{B} + \epsilon \boldsymbol{I})^{-1} \neq \boldsymbol{B}\dagger$ for all $\boldsymbol{B}$.

Proof: From an examination of eqns. 31 and 32 for $k = \hat{k}$, and since $\boldsymbol{B} \geqslant 0$, as $\epsilon \to 0$ the index $\bar{V}\{\boldsymbol{x}(\hat{k}), k_1, \hat{k}, \epsilon\}$ decreases monotonically. With the conditions given immediately after eqn. 2 or at the end of lemma 3 satisfied, the index $V\{\boldsymbol{x}(\hat{k}), k_1, \hat{k}, \epsilon\}$ has a lower bound for all $\epsilon \geqslant 0$ by the above corollary to lemma 3, and thus $\lim_{\varepsilon \to 0} \bar{V}\{\boldsymbol{x}(\hat{k}), k_1, \hat{k}, \epsilon\}$ exists for all $\boldsymbol{x}(k)$. With a control law $\bar{\boldsymbol{u}} = \boldsymbol{B}\dagger\, \boldsymbol{C}'\boldsymbol{x}$, and using eqn. 29,

$$\begin{aligned} V\{\boldsymbol{x}(k), \bar{\boldsymbol{u}}, k_1, \hat{k}, \epsilon\} &= \bar{V}\{\boldsymbol{x}(\hat{k}), k_1, \hat{k}\epsilon\} + \epsilon \bar{\boldsymbol{u}}'(\hat{k})\bar{\boldsymbol{u}}(\hat{k}) \\ &\geqslant \bar{V}\{\boldsymbol{x}(\hat{k}), k_1, \hat{k}, \epsilon\} \quad . \quad . \quad . \quad (33) \end{aligned}$$

From this equation, it may be concluded that

$$\bar{V}\{\boldsymbol{x}(\hat{k}), k_1, \hat{k}\} \geqslant \lim_{\varepsilon \to 0} \bar{V}\{\boldsymbol{x}(\hat{k}), k_1, \hat{k}, \epsilon\} \quad . \quad . \quad . \quad (34)$$

With a control law $\tilde{\boldsymbol{u}}$ which minimises the index $V\{\boldsymbol{x}(\hat{k}), \boldsymbol{u}, k_1, \hat{k}, \epsilon\}$, the following holds:

$$\begin{aligned} V\{\boldsymbol{x}(\hat{k}), \tilde{\boldsymbol{u}}, k_1, \hat{k}\} &= \bar{V}\{\boldsymbol{x}(\hat{k}), k_1, \hat{k}, \epsilon\} - \epsilon \tilde{\boldsymbol{u}}'(\hat{k})\tilde{\boldsymbol{u}}(\hat{k}) \\ &\geqslant \bar{V}\{\boldsymbol{x}(\hat{k}), k_1, \hat{k}\} \quad . \quad . \quad . \quad . \quad (35) \end{aligned}$$

From eqns. 34 and 35,

$$\lim_{\varepsilon \to 0} \bar{V}\{\boldsymbol{x}(\hat{k}), k_1, \hat{k}, \epsilon\} \geqslant \bar{V}\{\boldsymbol{x}(\hat{k}), k_1, \hat{k}\} \quad . \quad . \quad . \quad (36)$$

and thus, using eqns. 34 and 36,

$$\lim_{\varepsilon \to 0} \bar{V}\{\boldsymbol{x})(\hat{k}, k_1, \hat{k}, \epsilon\} = \bar{V}\{\boldsymbol{x}(\hat{k}), k_1, \hat{k}\}$$

or, equivalently, eqn. 32 is satisfied.

The above lemma is helpful in gaining another insight into the meaning of $\boldsymbol{\Pi}$. Note, however, that it has not been proved that $\lim_{\varepsilon \to 0} \tilde{\boldsymbol{u}} = \bar{\boldsymbol{u}}$.