A New Method for the Solution of the Cauchy Problem for Parabolic Equations

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An integral equation representation is given for parabolic partial differential equations. When the equations are defined in unbounded domains, as in the initial value (Cauchy) problem, the solution of the integral equation by the method of successive approximation has inherent advantages over other methods. Error bounds for the method are of order $h^{3/2}$ and $h^{7/2}$ (h is the increment size) depending on the finite difference approximations involved.

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Introduction

Parabolic partial differential equations [1] arise in most branches of science and engineering and a number of methods are now available to solve these equations numerically in bounded domains [2–5]. For the case where the differential equations are defined in unbounded domains, as in the initial value (Cauchy) problem, the obvious numerical approach to use is to approximate the solution by introducing a large boundary and boundary conditions, and solving the parabolic equation in the bounded domain using available methods. The difficulty, of course, is in choosing a bounded domain large enough to give a sufficiently good approximation to the desired result, and at the same time a boundary not so large that unnecessary computations are made.

In this paper, an integral equation is constructed, the solution of which, calculated using the method of successive approximations, approximates the desired parabolic equation solution. Error bounds are of the order of $h^{3/2}$ or $h^{7/2}$ (where h is the increment size) depending on the finite difference approximations used. The error introduced by using the method of successive approximations can be made arbitrarily small by taking a sufficiently large number of approximations. The step size h and the number of successive approximations determines a bounded domain which is used to approximate the unbounded one. From one viewpoint,

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the method is simply an efficient way of using a larger and larger approximating domain until a desired accuracy is achieved. It is inherently more efficient than choosing an appropriate approximating bounded domain by trial and error and at each trial applying existing methods (such as the method of lines) to the bounded domain problem.

In the following section, the various results are developed for the simplest of parabolic Cauchy problems. However, their extension to more general problems is straightforward. An illustration of the practical application of the results is given in [6], where the methods of this paper are used to solve an important problem in stochastic optimal control.

Integral Equation Representation and Error Bounds

The Cauchy problem considered here, in its simplest form, is the problem of finding a solution $u(x, t)$ of the parabolic equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x, t) \text{ in } \Omega \equiv R \times [0, T] \quad (1)$$

with initial condition $u(x, 0) = \phi(x)$ on R. Of course $f(x, t)$ is given throughout Ω, and to guarantee uniqueness of the solution $u(x, t)$, appropriate continuity and boundedness assumptions on f and ϕ are made [1]. In addition, we assume that

$$u(x) \in C^4[0, T], \quad |u(x)| \leq k_1 \exp k_2 |x|, \quad \left|\frac{\partial^4 u}{\partial x^4}(x)\right| \leq k_3 \exp(k_4 |x|)$$

for some constants k_1, k_2, k_3, k_4 and all $t \in [0, T]$.

For convenience, we will assume that a knowledge of $u(x, t)$ is required for $x = 0$, $t \in [0, T]$. (Actually, with only minor modifications to the following theory an arbitrary trajectory in Ω may be considered.) Integrating (1) yields the integral equation

$$u = \int_0^t \left[a^2 \frac{\partial^2 u}{\partial x^2} + f(x, \tau)\right] d\tau + \phi(x) \quad (2)$$

where it is understood that integration is along the $x = 0$ contour. Consider now a difference approximation to the partial deviative term as follows:

$$\frac{\partial^2 u}{\partial x^2} = \Delta^2 u - \frac{h^2}{12} \frac{\partial^4 \bar{u}}{\partial x^4} \quad (3)$$

where $\bar{u} = u(x \pm \beta h)$ for some $0 \leq \beta \leq 1$, and

$$\Delta^2 u \triangleq h^{-2}[u(x + h) - 2u(x) + u(x - h)] \quad (4)$$

for some constant h.

It is now meaningful to consider a finite difference approximation of (2) as

$$\hat{u} = \int_0^t [a^2 \Delta^2 \hat{u} + f(x, \tau)] \, d\tau + \phi(x) \quad (5)$$

where integration is along the path $x = 0$. We now show that this integral equation representation of (1) may be solved by the method of successive approximations.

First we restrict attention to a bounded domain $D \subset \Omega$ with $|x| \leq \bar{x}$ and give an error bound result for the error term $\tilde{u} = u - \hat{u}$ as

$$|\tilde{u}| \leq |a| h^{3/2} k_3 \exp(k_4 \bar{x})/24\beta\pi^2 \quad (6)$$

where $\beta = \frac{1}{2} - 2\pi/4$. This result is readily developed along the lines indicated in [4] for a closely related problem. Notice that the error term $|\tilde{u}|$ is of order $h^{3/2}$. (The related problem in [4] is in fact identical to the one considered here except that a higher order difference approximation is used, and as a consequence the error term in [4] is of order $h^{7/2}$. Since the result (6) is really incidental to the main idea of this paper, the derivations are not repeated here.)

One conclusion from (6) and the bound on $u(x)$ previously assumed as $|u(x)| \leq k_1 \exp(k_2 |x|)$ is that $\hat{u}$ is bounded as follows

$$\begin{aligned} |\hat{u}| &\leq |u| + |\tilde{u}| \\ &= k_1 \exp(k_2 \bar{x}) + (|a| h^{3/2}/24\beta\pi^2) k_3 \exp(k_4 \bar{x}) \\ &\leq k_5 \exp(k_6 \bar{x}) \end{aligned}$$

where $k_5 = k_1 + (|a| h^{3/2}/24\beta\pi^2) k_3$ and $k_6 = \max[k_2, k_4]$.

We now look more closely at the calculation of $\hat{u}$ by the method of successive approximations. Equation (5) may be written in operator notation as

$$\hat{u} = M\hat{u} \quad \text{with} \quad \|\hat{u}\| = \sup_D |\hat{u}|.$$

Observe that for u_0, an arbitrary function (e.g. $u = 0$) in D, we may write

$$M(\hat{u} - u_0) = \int_0^t a^2 \Delta^2 (\hat{u} - u_0) \, d\tau,$$

and by taking norms of both sides and using the bounds on $\hat{u}$, we can derive the following inequality:

$$|\hat{u} - Mu_0| \leq 4a^2 T k_5 \exp(k_6 \bar{x})/h^2.$$

This in turn may be readily generalized as

$$|\hat{u} - M^m u_0| \leq \frac{1}{m!} (4a^2 T k_5 \exp(k_6 \bar{x})/h^2)^m. \quad (7)$$

Clearly, for m large and $\bar{x} > (m + 1)h$, $M^m u_0$ is a close approximation to $\hat{u}$; that is, *M is a contraction operator*. Equivalently, $\hat{u}$ may be determined by the method of successive approximations.

The two results (6) and (7) now enable us to give an error bound for $|u - M^m u_0|$ if required, since

$$|u - M^m u_0| \leq |\hat{u} - M^m u_0| + |\tilde{u}|. \quad (8)$$

Clearly, for sufficiently large m the error is of order $h^{3/2}$.

The above results may be summarized as follows.

THEOREM. *Suppose there is given the parabolic equation*

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x, t) \text{ in } \Omega = R^n \times [0, T] \quad (1)$$

with initial condition $u(x, 0) = \phi(x)$ *on* R^n. *Suppose* *also that the solution u satisfies*

$$u(x) \in C^4[0, T], \quad |u(x)| \leq k_1 \exp(k_2|x|) \quad \left|\frac{\partial^4 u}{\partial x^4}(x)\right| \leq k_3 \exp(k_4|x|),$$

for some constants k_1, k_2, k_3, and k_4 for all $t \in [0, T]$. Then an integral equation representation of (1) for some constant h along any trajectory in Ω is

$$\hat{u} = \int_0^t \{(a/h)^2[\hat{u}(x+h) - 2\hat{u}(x) + \hat{u}(x-h)] + f(x,\tau)\}\, d\tau + \phi(x) \quad (5)$$

where bounds on the error $\tilde{u} = u - \hat{u}$ (see (6)) are of order $h^{3/2}$.

The integral equation (5) may be solved by successive approximations where the bounds on the error after m successive approximation is given in (7).

Some comments are in order.

1. More sophisticated difference schemes than indicated in (5) may be used to achieve error bounds of order $h^{7/2}$. For example, a scheme could be based on the difference approximation

$$\frac{d^2u(x+h)}{dx^2} + 10\frac{d^2u(x)}{dx^2} + \frac{d^2u(x-h)}{dx^2} - 12\Delta^2 u = R$$

where R is of order h^4.

2. Observe that for a particular integer m chosen for calulating $u(x, t)$ along a trajectory, $u(x \pm (m+2)h, t)$ is assumed equal to zero. With this boundary condition the solution $u(x, t)$ may be determined by any of the standard means for solving parabolic partial differential equations in bounded domains. Of course the solution near the boundary will have different error bounds than the solution along the specified trajectory.

3. The theorem above is readily generalized to parabolic equations.

$$\frac{\partial u}{\partial t} = \sum_{i,j}^{n} a_{ij}^2 \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i}^{n} b_i \frac{\partial u}{\partial x_i} + cu + f(x,t)$$

in $\Omega = R^n \times [0, T]$ with initial condition $u(x, 0) = \phi(x)$ in R^n. For this case, the coefficients, which may be functions of x and t, are required to be bounded as

$$|a_{ij}(x)| \leq \alpha_1 \exp(\alpha_2|x|),$$
$$|b_i(x)| \leq \alpha_3 \exp(\alpha_4|x|),$$
$$c_i(x) \leq \alpha_5 \exp(\alpha_6|x|)$$

for constants $\alpha_1 - \alpha_6$ and all $t \in [0, T]$. The bounds on the coefficients feature in the error bounds corresponding to (6) and (7) without affecting the key results; namely, that as m becomes large, the error (7) is small and the error (6) is of order $h^{3/2}$.

4. If the method above is applied to parabolic problems defined in bounded regions with, say, $x \in [0, 1]$, then a vector $U = [u(0), u(h), u(2h), \ldots, u(1)]$ may be defined and an integral equation can be written for U. However, solving this equation by the method of successive approximations could be less efficient than solving the corresponding differential equation using standard techniques—as is done in the method of lines [3, 4].

5. In the practical application of the method of successive approximations, the number of approximations to take is not determined from the error bounds [7], but simply from the results at each step in the iterative procedure using standard methods.

6. In a companion paper [6], the above theorem is applied to an important problem in stochastic control theory. In [6], important cases are studied when it is required to solve *elliptic* partial differential equations:

$$0 = \sum_{i,j}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i}^{n} b_i \frac{\partial u}{\partial x_i} + cu + f$$

defined in unbounded domains with u specified on a same domain S. This elliptic equation can be reduced to a parabolic equation along trajectories

$$\frac{d}{d\tau} x_i(\tau) = b_i(x(\tau), \tau) \text{ with } x(t) = x$$

when these terminate on S in a finite time. The parabolic equations are

$$\frac{du}{dt} = \sum_{i,j}^{n} a_{ij}^2 \frac{\partial^2 u}{\partial x_i \partial x_j} + cu + f$$

with initial conditions determined from the boundary conditions above. Difficulties arise due to the fact that u is unspecified for infinite x. These are resolved in [6], and the required solution is determined using finite difference approximations and the method of successive approximations of this paper.

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