Fixed-Lag Smoothing for Nonlinear Systems with Discrete Measurements*

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ABSTRACT

Approximate nonlinear filtering formulas are applied to yield fixed-lag smoothing results for nonlinear systems with discrete noisy observations. A signal process model is defined which allows the required fixed-lag estimates to be obtained as filtered estimates. The smoothing results are only approximations since the nonlinear filtering theory used is based on simplifying approximations, but assuming that the simplifications are valid, then the fixed-lag smoothed estimate is a better one than simply a filtered estimate. In some circumstances the improvement may be quite substantial.

INTRODUCTION

Non-linear estimation theory ranges from equations governing the exact evolution of the probability density of the quantities to be estimated conditioned on the measurements, see for example Refs. 1-3, to different algorithms for calculating an approximate estimate and the associated error covariance [1, 4-7]. Filtering theory results may be extended to cover the cases of prediction, fixed point smoothing, fixed interval smoothing, and, as shown in this paper, fixed-lag smoothing.

The advantage of fixed-lag smoothing when compared to filtering is that improved estimates (equivalently lower error covariances) are achieved, and in many cases these improvements may be of considerable significance. The disadvantage, of course, is that there is a fixed delay between observations associated with a signal and its estimation which does not exist for the filtering case. There are many applications in both communications and control where a small delay is quite acceptable, although, of course, there are some applications, such as in feedback control of high gain systems, where a small delay in estimation would be intolerable. The advantage of fixed-lag smoothing when compared to the

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other two forms of smoothing is simply that estimates can be made "on-line," and if the fixed-lag is of the order of two or three times the dominant time constant of the signal process model, the fixed-lag smoothed estimates are for most purposes as good as those from fixed-interval or fixed-point smoothing.

The emphasis in this paper is on practical fixed-lag smoothing algorithms and so the theory for the exact evolution of the conditional probability density for the fixed-lag smoothing case will not be considered.

The approach taken in the paper may be outlined as follows. Consider that noisy measurements $z(t_k)$ of the state $x(t_k)$ of a signal process model are received, where $k = 0, 1, \ldots$. The conditioned mean filtered estimates $\hat{x}(t_k|t_k)$ and fixed-lag smoothed estimates $\hat{x}(t_{k-N}|t_k)$ are defined from

$$\hat{x}(t_k|t_k) = E[x(t_k)|Z(t_k)],$$

$$\hat{x}(t_{k-N}|t_k) = E[x(t_{k-N})|Z(t_k)],$$

where N is the value of the fixed lag and $Z(t_k)$ is the sequence $z(t_0)$, $z(t_1)$, $\ldots, z(t_k)$. Consider also the state delayed by the amount of the fixed-lag N denoted $x_N(t_k) = x(t_{k-N})$ and its filtered estimate $\hat{x}_N(t_k|t_k)$. Observe that

$$\begin{aligned}\hat{x}(t_{k-N}|t_k) &= E[x(t_{k-N})|Z(t_k)],\\ &= E[x_N(t_k)|Z(t_k)],\\ &= \hat{x}_N(t_k|t_k).\end{aligned}$$

In other words, the optimal fixed-lag smoothed estimate of the state of a signal process is simply the optimal filtered estimate of the state delayed by the fixed-lag. This means that by an appropriate augmentation of the signal process model to include the states $x_N(t_k)$, filtering theory can be applied to the augmented model to achieve an optimal filter which has as its output $\hat{x}_N(t_k|t_k)$. This filter is in fact the required optimal fixed-lag smoother since an alternative formulation of the output is $\hat{x}(t_{k-N}|t_k)$.

The approach just described has been explored in a companion paper for the linear discrete fixed-lag smoothing case [8]. The extension to the nonlinear case when the signal process model is purely discrete is reasonably straightforward—the essentially new ingredients being the application of approximate formulas and interpretations of the results. On the other hand, the case which may be encountered in practice of a *continuous nonlinear system with discrete observations* requires an essentially different form of signal process model than has been suggested for the corresponding linear problem. Moreover, in order to apply nonlinear filtering theory to the augmented signal process model proposed in this paper, a slightly more general filtering theory is required than that usually expounded in the literature. Albeit, the smoothing results developed in this paper can be specialized to the case of linear continuous systems with discrete noisy measurements, or to discrete nonlinear systems with discrete noisy measurements, in which case the derivations represent alternative ones to those proposed in the companion paper on linear discrete fixed-lag smoothing results.

The outline of the paper is as follows. In the next section the equations for one of the approximate nonlinear filtering algorithms are reviewed in detail while only passing reference is made to other algorithms, in order to keep the number of equations used to explain the key ideas of this paper to a minimum. The approximate algorithm chosen is that known as the modified truncated second order type. In Sec. 3, this algorithm is applied to an augmented signal process model to yield approximate optimal fixed-lag nonlinear smoothing results.

2. REVIEW OF FILTERING FORMULAS

Consider the case of a continuous nonlinear n-dimensional system with discrete observations described by the state equations

$$\dot{x}(t) = f(x, t) + G(x, t)u(t), \tag{1}$$

$$z(t_{k+1}) = h[x(t_{k+1}), t_{k+1}] + v(t_{k+1}), \tag{2}$$

where $t_k \leqslant t < t_{k+1}$ for $k = 0, 1, \ldots$. The initial state $x(t_0)$ is assumed to be a gaussian random variable with mean m_0 and covariance $P(t_0)$. The noise vectors $u(\cdot)$ and $v(\cdot)$ are independent zero mean white gaussian processes with

$$E\{u(t)u'(\tau)\} = I(t)\delta(t-\tau), \text{ and } E\{v(t_k)v'(t_l)\} = R(t_k)\delta_{kl}$$

The above equations, (1) and (2), are those normally used for a signal process model for the case of a continuous nonlinear system with discrete noisy measurements. In this section we consider (1) and (2) together with the additional equations

$$x(t_{k+1}) = \Phi(t_{k+1})\, x(t_{k+1}^-) \tag{3}$$

representing discontinuities at t_k. The notation t_k^- is used to denote $t_{k-\epsilon}$ for arbitrary small $\epsilon > 0$. Of course we may secure the usual filtering results by setting $\Phi = I$. The inclusion of Eq. (3) allows us to apply filtering results to achieve a fixed-lag smoother as the next section shows. (Actually, an even more general equation than Eq. (3) could be considered, namely $x(t_{k+1}) = \Phi[x(t_{k+1}^-), t_{k+1}] + g_1[x(t_{k+1}^-), t_{k+1}]\, u_1(t_{k+1})$. Using such an extension would allow specialization of the results to purely discrete systems by setting f and G in Eq. (1) equal to zero.

Numerous approximate nonlinear filtering algorithms have been proposed. The simplest and to date the most useful is the extended Kalman filter. The "modified second order truncated type" is a further improvement on this filter and will serve as an example of an approximate nonlinear filtering algorithm for the remainder of the paper, although passing reference will be made to other algorithms.

MODIFIED TRUNCATED SECOND-ORDER APPROXIMATE NONLINEAR FILTER

The equations fall into two categories—those between observations and those at observation instants.

Between Observations

$t_k \leqslant t < t_{k+1}$ for $k = 0, 1, \ldots$. The predicted estimate $\hat{x}(t|t_k)$ is given from the two coupled equations

$$\dot{\hat{x}}(t|t_k) = f(\hat{x}, t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\hat{x}, t) : P(t|t_k), \tag{4}$$

$$\dot{P}(t|t_k) = F(t)P(t|t_k) + P(t|t_k)F'(t) + \widehat{G(\hat{x}, t)G'(\hat{x}, t)}, \tag{5}$$

where the notation is as follows

$$\frac{\partial^2 f(\hat{x}, t)}{\partial x^2} : P(t|t_k) = \sum_{i,j=1}^{n} P_{ij}(t|t_k) \left.\frac{\partial^2 f(x, t)}{\partial x_i \partial x_j}\right|_{x=\hat{x}} \tag{6}$$

$$F_{ij}(t) = \left.\frac{\partial f_i(x, t)}{\partial x_j}\right|_{x=\hat{x}} \tag{7}$$

For the truncated second order algorithm

$$\widehat{G(\hat{x}, t)G'(\hat{x}, t)} = G(\hat{x}, t)G'(\hat{x}, t) + \frac{1}{2}\frac{\partial^2 G(\hat{x}, t)G'(\hat{x}, t)}{\partial x^2} : P(t|t_k). \tag{8}$$

At Observation Instants

t_k for $k = 0, 1, \ldots$ following from Eq. (2) we have

$$\hat{x}(t_{k+1}|t_k) = \Phi(t_{k+1})\hat{x}(t_{k+\bar{1}}|t_k), \tag{9}$$

$$\begin{aligned} P(t_{k+1}|t_k) &= E\{[x(t_{k+1}) - \hat{x}(t_{k+1}|t_k)]\,[x(t_{k+1}) - \hat{x}(t_{k+1}|t_k)]'\}, \\ &= \Phi(t_{k+1})P(t_{k+\bar{1}}|t_k)\Phi'(t_{k+1}). \end{aligned} \tag{10}$$

The relevant approximate nonlinear filtering equations at the observation instants are as follows:

$$\hat{x}(t_{k+1}|t_{k+1}) = \hat{x}(t_{k+1}|t_k) + P(t_{k+1}|t_k)H(t_{k+1})Y^{-1}(t_{k+1})\tilde{z}(t_{k+1}), \tag{11}$$

$$P(t_{k+1}|t_{k+1}) = P(t_{k+1}|t_k) - P(t_{k+1}|t_k)H(t_{k+1})Y^{-1}(t_{k+1})H'(t_{k+1})P(t_{k+1}|t_k), \tag{12}$$

where $\hat{x}(t_{k+1}|t_k)$ and $P(t_{k+1}|t_k)$ are calculated from Eqs. (9) and (10) and the following definitions apply for H, $\tilde{z}$, and Y.

$$H'_{ij}(t_{k+1}) = \left.\frac{\partial h_i(x, t_{k+1})}{\partial x_j}\right|_{x(t_{k+1})=\hat{x}(t_{k+1}|t_k)} \tag{13}$$

$$\tilde{z}(t_{k+1}) = z(t_{k+1}) - h[\hat{x}(t_{k+1}|t_k), t_{k+1}] - \frac{1}{2}\frac{\partial^2 h}{\partial x^2}[\hat{x}(t_{k+1}|t_k), t_{k+1}] : P(t_{k+1}|t_k) \tag{14}$$

$$Y(t_{k+1}) = H'(t_{k+1})P(t_{k+1}|t_k)H(t_{k+1}) + R(t_{k+1}) - \frac{1}{4}\left[\frac{\partial^2 h}{\partial x^2}[\hat{x}(t_{k+1}|t_k), t_{k+1}] : P(t_{k+1}|t_k)\right]\left[\frac{\partial^2 h'}{\partial x^2}[x(t_{k+1}, t_k), t_{k+1}] : P(t_{k+1}|t_k)\right] \tag{15}$$

Notice that the only additions to nonlinear filtering theory caused by introducing Eq. (3) are Eqs. (9) and (10). Setting $\Phi = I$ in the above equations (the usual filtering case) eliminates the need for Eqs. (9) and (10).

OTHER APPROXIMATE FILTERS

The extended Kalman filter results can be derived as a specialization of the above equations—the second order derivatives are simply set to zero.

The regular truncated second order approximate nonlinear filter is as given in the above equations except that additional terms are included in calculating $P(t_{k+1}|t_{k+1})$ in Eq. (12).

The modified Gaussian second order approximate nonlinear filter requires additional terms in the approximate evaluation of $\widehat{GG'}$ given in Eq. (8), and the term Y in Eq. (15) is defined a little differently. The regular Gaussian second order filter requires the additional terms in Eq. (12). The iterated extended Kalman filter or iterated versions of any of the approximate filters for that matter as the names suggest, require an iteration or two on the various equations to achieve consistence. It will be observed that the various additional terms and procedures mentioned do not affect the ideas to follow in the next section.

3. APPROXIMATE NONLINEAR FIXED-LAG SMOOTHERS

In this section the filtering theory of the previous section is applied to an augmented signal process model to yield fixed-lag smoothing results. The key step is the first step which is to arrive at the appropriate signal process model to which filtering ideas may be applied.

At first glance it might be thought that augmenting the continuous part of the original signal process model with delay elements equal to the fixed-lag would be appropriate since, as indicated in Sec. 1, filtered estimates of the states delayed by the amount of the fixed-lag are, in fact, fixed-lag smoother estimates of the states. This is all well and good except that the filtering theory can not readily be applied to signal process models with delay elements. By using a finite dimensional approximation for the representation of the delay elements an arbitrarily good approximation could be achieved from an application of filtering theory to this model. The difficulty with this approach is in the first step of selecting a suitable finite dimensional approximation to the delay.

Further thought shows that there are in fact two problems that can be isolated and solved sequentially rather than simultaneously. The first problem is to achieve fixed-lag estimates at the observation instants, and the second problem is to achieve values for the fixed-lag estimates between the observations. The formulation of a finite dimensional augmented signal process model so that the filtering theory of the preceding section can be applied to achieve a solution to the first problem can be achieved. The details are now developed.

(a) SMOOTHED ESTIMATES AT THE OBSERVATION INSTANTS.

As a first step to achieving smoothed estimates at the observation instants, a finite dimensional augmented signal process model is constructed such that the original system states together with delayed values of these are available at the observation instants. That is, the model state vector at time t_k should include $x(t_k)$ and $x(t_{k-N})$ where the fixed-lag is N intervals between observations. Such a model is given by the following equations between observations with $t_k \leqslant t < t_{k+1}$

$$\begin{bmatrix} \dot{x} \\ \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} f(x,t) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} G(x,t) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(t), \tag{16}$$

and at observations t_{k+1}

$$\begin{bmatrix} x(t_{k+1}) \\ x_1(t_{k+1}) \\ \vdots \\ x_N(t_{k+1}) \end{bmatrix} = \left[\begin{array}{cccc|c} I & 0 & \cdot & \cdot & 0 \\ \hline & & & & \\ & & I & & \vdots \\ & & & & 0 \end{array}\right] \begin{bmatrix} x(t_{k+\bar{1}}) \\ x(t_{k+\bar{1}}) \\ \vdots \\ x_N(t_{k+\bar{1}}) \end{bmatrix} \tag{17}$$

$$z(t_{k+1}) = h[x(t_{k+1}), t_{k+1}] + v(t_{k+1}), \tag{18}$$

for $k = 0, 1, \ldots$. The solution of Eq. (16) yields $x_i(t) = x_i(t_k)$ for $t_k \leqslant t < t_{k+1}$ and in particular

$$x_i(t_{k+\bar{1}}) = x_i(t_k)$$

for $i = 1, 2, \ldots N$. Substitution of this equation into Eq. (17) yields

$$x_i(t_{k+1}) = x_{i-1}(t_{k+\bar{1}}) = x_{i-1}(t_k) \cdots = x(t_{k+1-i})$$

for $i = 1, 2, \ldots N$. The notation $x(\cdot) = x_0(\cdot)$ is adopted for convenience. Of particular interest for us here is the case $i = N$, for which

$$x_N(t_{k+1}) = x(t_{k+1-N}) \tag{19}$$

We conclude that the signal process model, (16) and (17), does in fact have a state vector at time t_k which includes both $x(t_k)$ and $x(t_{k-N}) = x_N(t_k)$ as required, as well as conforming to the finite dimensionality required for application of the filtering theory of the previous section.

Application of the filtering theory of the previous section to the augmented signal process model, (16) and (17), is now straightforward.

The approximate error covariance vector can be conveniently partitioned as

$$\begin{bmatrix} P_{00} & P'_{10} & \cdots & P'_{N_0} \\ P_{10} & P_{11} & & \\ \vdots & & & \\ P_{N_0} & & & P_{NN} \end{bmatrix} = \begin{bmatrix} P & P'_1 & \cdots & P'_N \\ P_1 & P_{11} & & \\ \vdots & & & \\ P_N & & & P_{NN} \end{bmatrix}$$

where, as indicated, subscripts that are zero are deleted to simplify notation. The filtering equations for this case appear formidable when written out in full, but simplify immediately because of the many zero entries in the various matrices. They consist of the original filter equations as expected with additional equations.

$$\dot{\hat{x}}_i(t|t_k) = 0, \tag{20}$$

$$\dot{P}_i(t|t_k) = P_i(t|t_k)F(t_k), \tag{21}$$

$$\dot{P}_{ii}(t|t_k) = 0, \tag{22}$$

$$\hat{x}_i(t_{k+1}) = \hat{x}_{i-1}(t_{k+\bar{1}}), \tag{23}$$

$$P_i(t_{k+1}|t_k) = P_{i-1}(t_{k+\bar{1}}|t_k), \tag{24}$$

$$P_{ii}(t_{k+1}|t_k) = P_{i-1,\,i-1}(t_{k+\bar{1}}|t_k), \tag{25}$$

$$\hat{x}_i(t_{k+1}|t_{k+1}) = \hat{x}_i(t_{k+1}|t_k) + P_i(t_{k+1}|t_k)H(t_{k+1})Y^{-1}(t_{k+1})z(t_{k+1}), \tag{26}$$

$$P_i(t_{k+1}|t_{k+1}) = P_i(t_{k+1}|t_k) - P_i(t_{k+1}|t_k)H(t_{k+1})Y^{-1}(t_{k+1})H'(t_{k+1})P(t_{k+1}|t_k), \tag{27}$$

$$P_{ii}(t_{k+1}|t_{k+1}) = P_{ii}(t_{k+1}|t_k) - P_i(t_{k+1}|t_k)H(t_{k+1})Y^{-1}(t_{k+1})H'(k+1)P_i(t_{k+1}|t_k). \tag{28}$$

Where $i = 1, 2, \ldots N$, $t_k < t < t_{k+1}$, $k = 0, 1, \ldots$ and F, H, Y, and $\tilde{z}$ are as defined for the filter in Sec. 2. (The case P_{ij} for $i \neq 0, i \neq j$, is not necessary for the theory of this section.)

In view of Eqs. (11) and (12), we can express $HY^{-1}\tilde{z}$ and $HY^{-1}H'$ in terms of the filter estimates $\hat{x}$ and error covariances P. Thus, Eqs. (26)-(28) can be replaced by the following set of equations for the fixed-lag smoothing:

$$\hat{x}_i(t_{k+1}|t_{k+1}) = \hat{x}_i(t_{k+1}|t_k) + P_i(t_{k+1}|t_k)P^{-1}(t_{k+1}|t_k)[\hat{x}(t_{k+1}|t_{k+1}) - \hat{x}(t_{k+1}|t_k)], \tag{29}$$

$$P_i(t_{k+1}|t_{k+1}) = P_i(t_{k+1}|t_k) - P_i(t_{k+1}|t_k)[I - P^{-1}(t_{k+1}|t_k)P(t_{k+1}|t_{k+1})], \tag{30}$$

$$P_{ii}(t_{k+1}|t_{k+1}) = P_{ii}(t_{k+1}|t_k) - P_i(t_{k+1}|t_k)P^{-1}(t_{k+1}|t_k) \cdot [I - P(t_{k+1}|t_{k+1})P^{-1}(t_{k+1}|t_k)]\, P_i(t_{k+1}|t_{k+1}). \tag{31}$$

Thus the smoothing equations do not depend on the particular form of Y and $\tilde{z}$ of the filter. Hence Eq. (20)-(25) and (29)-(31) can be used in conjunction with some other types of approximate filters such as the modified gaussian second-order filter and the extended Kalman filter, as long as the filtering equations are of the same form as Eqs. (11) and (12). Also, note that the term $\widehat{GG'}$ does not appear in the additional equations for smoothing and so its definition is irrelevant to the design of the smoothers once the filter has been designed.

In summary, filter equations (for example Eqs. (4)-(15) of the previous section) together with Eqs. (20)-(25) and (29)-(31) of this section are the required filtering equations of the augmented system (16) and (17), and since from Eq. (19)

$$\hat{x}_N(t_{k+1}|t_{k+1}) = \hat{x}(t_{k+1-N}|t_{k+1}). \tag{32}$$

These equations together with Eq. (32) yield the required fixed-lag smoothing equations. The matrix $P_{NN}(t_{k+1}|t_{k+1})$ is, in fact, the error covariance of the fixed-lag smoother. It is not difficult to show from a successive application of Eq. (28) that $[P(t_{k+1}|t_{k+1}) - P_{NN}(t_{k+1}|t_{k+1})]$ the improvement in error due to smoothing rather than filtering is always non-negative, and in many instances is significantly large. Further results in this area can be developed as for the linear signal model case. Suffice it to say here that we expect that the larger the fixed-lag the greater the improvement, although there will be some finite lag period beyond which the improvement is negligible. As a guide on this, for the linear case this lag period is of the order of two or three time constants of the filter.

The dimension of the fixed-lag smoother as just developed is $n(N + 1)$. Reduced order smoothers are possible for the case of signal smoothing rather than state smoothing. That is, for the case where a smoothed estimate of $h(x, t)$ is required rather than of x itself, and the dimension of $h(x, t)$ is less than that of x, reduced order estimators are possible. The only change is for the augmented integrators to have as their state vectors $x_i(t) = h[x_{i-1}(t_k^-), t_k]$ rather than $x_i(t) = x_{i-1}(t_k^-)$ for $t_k \leqslant t < t_{k+1}$ and $i = 1, 2, \ldots N$. Other possibilities of reduced order smoothers also exist in special cases as illustrated by those given for the linear case in the companion paper [8].

(b) SMOOTHED ESTIMATES BETWEEN OBSERVATIONS.

The augmented signal process model just considered is not constructed to yield optimal smoothed estimates between observations. Notice, however, from Eq. (20) that $x_N(t|t_k) = x_N(t_k|t_k)$ for $t_k \leqslant t < t_{k+1}$. In other words, the theory to date yields an estimate between observations which is simply a constant value, namely, the smoothed estimates at the previous observation instant. This in many situations may be adequate.

To obtain smoothed data at all points between observation instants is a difficult problem, but to obtain this data at a finite number of points between observation instants does not require more than an almost trivial extension to the theory so far presented, as we now indicate.

Consider that we artifically introduce "observation" instants midway between our actual observation instants. Let us denote this instant by t_k where $k = ½, 1½, 2½, \ldots$. The theory developed in the first part of this section can be applied directly to this modified model to yield smoothed estimates at t_k for $k = 0, ½, 1, 1½ \ldots$. Notice that for a fixed time lag the number of "observation" intervals has doubled and the order of integrators has doubled! This idea can of course be applied for an arbitrary finite number of artifically introduced "observation" instants between our actual observations.

4. CONCLUSION

We have shown that approximate fixed-lag smoothing of discrete noisy measurements of a continuous time nonlinear stochastic process can be achieved using known approximate nonlinear filtering algorithms or rather slight generalizations of these.

One area for application investigated in Ref. 9 is the optimal fixed-lag demodulation of discrete noisy measurements of FM signals. Perhaps the simplest problem in this area of application is the fixed-lag smoothing of nonlinear noisy measurements of a continuous time stationary stochastic process. For this problem $f(x, t) = Fx$, $\partial^2 f/\partial x^2 = 0$ and thus the equations are considerably simplified. For the case where there is channel memory such simplifications are not possible and the full power of the present theory is required.

Another area of investigation is the optimal fixed-lag demodulation of pulse frequency modulated (PFM) signals [10]. These particular applications are readily seen to be cases where a small delay in arriving at an optimal estimate is perfectly acceptable, particularly if the estimate is a significantly better one than where no delay is used.

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