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Brief Paper

Fixed-Lag Demodulation of Discrete Noisy Measurements of FM Signals*

Démodulation à Délai Fixe de Mesures de Signaux FM Bruyants Discrets

Demodulation mit fester Verzögerung bei diskreten Rauschmessungen von FM—Signalen

Демодуляция с постоянным запаздыванием для дискретных измерений шумов в ЧМ сигналах

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Summary—This paper considers an application of some recent results of fixed-lag smoothing algorithms to the demodulation of discrete noisy measurements of FM signals. The problem is attacked by first modelling the communications system in terms of a finite dimensional nonlinear stochastic system and relating the resulting model to an estimation problem for which approximate solutions can be obtained. Performance results are presented, using the modified truncated second order filtering algorithm. The results illustrate that if delay is allowed in arriving at an "on-line" optimum estimate, significant improvement can be obtained by even a very moderate fixed-lag.

1. *Introduction*

In this paper, we aim at achieving two goals simultaneously:

(1) To demonstrate an approach of obtaining improved FM demodulation. The approach is quasi-optimum fixed-lag smoothing of the received signal after sampling.

(2) To provide some numerical results in the area of nonlinear fixed-lag smoothing theory.

Stochastic estimation theory has been applied to the demodulation problem by various authors [1–5, 14–16]. It is found that the quasi-optimum FM demodulator when simplified is the familiar phase-locked loop used for high carrier to noise ratio (CNR) [5–7].‡ With the advancement of digital techniques, many investigators considered the problem of digital realizations of the analog phase-locked loop e.g. [8–10]. In particular, Kelly and Gupta [4, 14] considered the discrete time demodulation of continuous time signals using state space techniques. The resulting digital FM demodulators possess performance characteristics that match those of the analog quasi-optimum realizable demoulator.

It is well known that improved demodulation performance can be achieved using demodulation with delay. This point is well illustrated in Fig. 5.27 of Ref. [5]. Continuous time demodulation with delay requires the realization of a pure time delay equal to the value of the delay and is for many situations an unattractive proposition [13]. In this paper, we consider discrete-time demodulation of the received signal after sampling and point out that demodulation with delay and thus improved performance can be achieved in a straightforward manner. The results are a natural extention of those of Ref. [4] using the fixed-lag smoothing formulas of Ref. [11].

2. *Review of fixed-lag smoothing formulas developed in* [11]

Consider the case of a continuous nonlinear n-dimensional system with discrete observations described by the state equations

$$\dot{x}(t)=f(x,\ t)+G(x,t)\ u(t) \tag{1}$$

$$z(t_{k+1})=h[x(t_{k+1}),\ t_{k+1}]+v(t_{k+1}) \tag{2}$$

where $t_k \leq t < t_{k+1}$ for $k=0, 1, \ldots$. The initial state $x(t_0)$ is assumed to be a Gaussian random variable with mean m_0 and covariance $P(t_0)$. The noise vectors $u(\cdot)$ and $v(\cdot)$ are independent zero mean white Gaussian processes with

$$E\{u(t)u'(t)\}=I(t)\delta(t-\tau) \text{ and } E\{v(t_k)v'(t_l)\}=R(t_k)\delta_{kl}.$$

The above equations (1) and (2) are those normally used for a signal process model for the case of a continuous nonlinear system with discrete noisy measurements.

Numerous approximate nonlinear filtering algorithms have been proposed, e.g. see [12]. The simplest and to date the most useful is the extended Kalman filter. The "modified second order truncated type" is a further improvement on this filter and will serve as an example of an approximate nonlinear filtering algorithm described in [12] for our demodulator example here.

Smoothed estimates. It is shown in [11] that the required smoothing formulas consists of the original filter equations with the additional equations.

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§ For more complete bibliography of this subject, refer to chapter 3 of Ref. [5].

$$ {}_i(t|t_k)=0 \quad (3)$$

$$\dot{P}_i(t|t_k)=P_i(t|t_k)F(t) \quad (4)$$

$$\dot{P}_{ii}(t|t_k)=0 \quad (5)$$

$$\hat{x}_i(t_{k+1})=\hat{x}_{i-1}(t_{k+1}^-) \quad (6)$$

$$P_i(t_{k+1}|t_k)=P_{i-1}(t_{k+1}^-|t_k) \quad (7)$$

$$P_{ii}(t_{k+1}|t_k)=P_{i-1,\,i-1}(t_{k+1}^-|t_k) \quad (8)$$

$$\hat{x}_i(t_{k+1}|t_{k+1})=\hat{x}_i(t_{k+1}|t_k)+P_i(t_{k+1}|t_k)P^{-1}(t_{k+1}|t_k)[\hat{x}(t_{k+1}|t_{k+1})-\hat{x}(t_{k+1}|t_k)] \quad (9)$$

$$P_i(t_{k+1}|t_{k+1})=P_i(t_{k+1}|t_k)-P_i(t_{k+1}|t_k)[I-P^{-1}(t_{k+1}|t_k)P(t_{k+1}|t_{k+1})] \quad (10)$$

$$P_{ii}(t_{k+1}|t_{k+1})=P_{ii}(t_{k+1}|t_k)-P_i(t_{k+1}|t_k)P^{-1}(t_{k+1}|t_k)[I-P(t_{k+1}|t_{k+1})P^{-1}(t_{k+1}|t_k)]P_i(t_{k+1}|t_{k+1}). \quad (11)$$

Where

$$i=1,\,2,\,\ldots\,N,\; x_i(t_k)\equiv x(t_{k-i}),\; P_i(t|t_k)\equiv E\{[x_i(t)-\hat{x}_i(t|t_k)][x(t)-\hat{x}(t|t_k]'\}P_{ii}(t|t_k)\equiv E\{[x_i(t)-\hat{x}_i(t|t_k)][x_i(t)-\hat{x}_i(t|t_k)]'\},\; t_k\leq t<t_{k+1},\; k=0,\,1,\,\ldots \text{ and}$$

$$F^{(i,\,j)}(\mathrm{t})=\left.\frac{\partial f^{(i)}(x,\,t)}{\partial x^{(j)}}\right|_{x=\hat{x}},$$

where $F^{(i,j)}(\mathrm{t})$ denotes the (i,j)th element of the matrix F and $f^{(i)}$ denotes the ith component of the vector f.

Equations (3)–(11) can be used in conjunction with some other types of approximate filters such as the modified Gaussian second-order filter and the extended Kalman filter.

3. *Fixed-lag demodulation of FM signals using discrete measurements*

The problem considered in Ref. [4] is an approach to the digital demodulation of an analog communications signal with a minimum mean square error criterion of optimality. The communications problem is cast in terms of a state estimation problem similar to the one of SNYDER [3]. It is found that the modified truncated second-order filter worked reasonably well for the FM demodulation problem considered. Our demodulation example here will be the scalar message model considered in Section 3 of Ref. [4]. The relevant equations to be considered are:

$$\dot{x}(t)=\begin{bmatrix}\dot{\lambda}(t)\\ \dot{\psi}(t)\end{bmatrix}=\begin{bmatrix}-a & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}\lambda(t)\\ \psi(t)\end{bmatrix}+\begin{bmatrix}\sqrt{2a}\\ 0\end{bmatrix}u(t) \quad (12)$$

$$h(t)=\sqrt{2}\,\sin\,(W_c t+\delta\psi(t)) \quad (13)$$

$$z(t_k)=h(t_k)+v(t_k) \quad (14)$$

where

$$E[u(t)u(\tau)]=q\delta(t-\tau) \quad (15)$$

$$E[v(t_k)v(t_l)]=R(t_k)\delta_{kl} \quad (16)$$

$$E[v(t_k)u(\tau)]=0. \quad (17)$$

Here $\lambda(t)$ is the scalar message to be estimated using the scalar noisy measurements $z(t_k)$.

It is shown in Ref. [14] that the modified truncated second-order filter can be applied to (12) to yield a demodulator which consists of a digital phase-locked loop.

It is obvious that equations (3)–(11) for fixed-lag smoothing can be applied to this demodulator which we shall call "zero-lag" demodulator. For the demodulator using a fixed-lag of N sampling intervals, we shall call it an "N-lag" demodulator. The performance of the digital demodulator is expressed by the output signal-to-noise ratio (SNR) which is given by Ref. [4]:

$$\mathrm{SNR}(t)=\frac{1}{P_\lambda(t|t)}-1\ (\text{for } E[\lambda^2]=1) \quad (18)$$

where

$$P_\lambda(t|t)=E[(\lambda-\hat{\lambda}(t|t))^2] \quad (19)$$

The output signal-to-noise ratio depends on the following four design factors [14]:

(1) bandwidth expansion factor $\beta=\sqrt{q\delta/a}$;

(2) noise bandwidth $B=2\theta\beta a/2\pi$ of the IF filter, where θ is to be chosen by the designer;

(3) sampling interval $T\leq 1/B$;

(4) carrier-to-noise ratio (CNR) in the message bandwidth.

Performance results of the demodulator are shown in Figs. 1 and 2. These are curves of SNR plotted against CNR for particular values of β, θ and T. The uppermost curve in each diagram gives the inverse of mean-square unrealizable error for FM systems, as is given in Fig. 4.3 of [5]. This presents the uppermost limit of the performance of an FM demodulator. The diagrams show that our fixed-lag demodulator do approach this uppermost limit with a lag of only several sampling intervals.

At this point, we wish to review our somewhat "ideal" FM system and indicate what we should do if some of the assumptions are removed.

We have assumed a linear message model. What happens if the message model is nonlinear? Again, even if the message model is linear, what happens if the message is distorted by some nonlinear transformation, introduced either intentionally or unintentionally, before being frequency modulated at the transmitter? The answer is: our filtering and

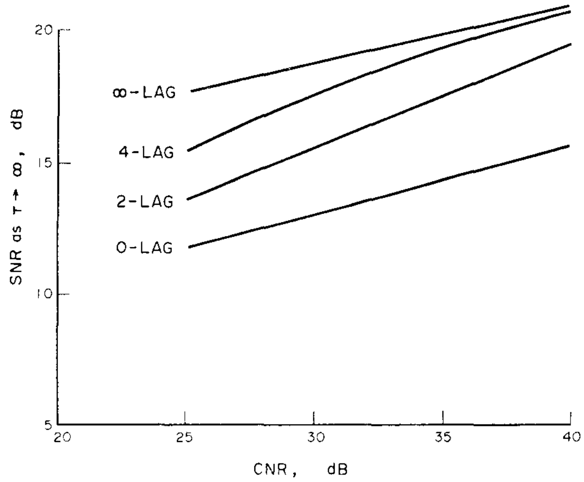

FIG. 1. FM demodulator performance for $\beta=100$ $\theta=5/T=0\cdot1$ (SNR output as a function of CNR input).

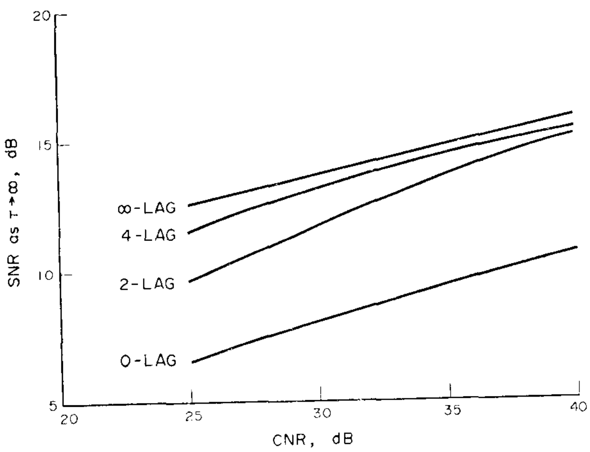

FIG. 2. FM demodulator performance for $\beta=10$ $\theta=5/T=0\cdot5$ (SNR output as a function of CNR input).

smoothing equations of Section 2 still apply. We only need to replace equation (12) by more involved equations. As an illustration we consider some sort of compression of the message is introduced before being frequency modulated, as shown in Fig. 3. Then equation (12) has to be replaced by the following equations:

$$\dot{x}=\begin{bmatrix}\lambda\\ \psi\end{bmatrix}=f(x)+Gu=\begin{bmatrix}-a\lambda\\ \lambda'\end{bmatrix}+\begin{bmatrix}\sqrt{2a}\\ 0\end{bmatrix}u \quad (20)$$

where

$$\lambda'=\alpha\log\left(1+\frac{\mu\lambda}{V}\right) \quad o\leq\lambda\leq V \quad (21)$$

$$-\alpha\log\left(1-\frac{\mu\lambda}{V}\right) \quad -V\leq\lambda\leq o$$

$$\alpha=\frac{V}{\log(1+\mu)} \quad (22)$$

and μ, V are constants.

Then we have

$$\dot{\hat{x}}(t|t_k)=f(\hat{x}(t|t_k))+(0\cdot5)P^{(1,1)}(t|t_k)\begin{bmatrix}0\\ \dfrac{\partial^2\hat{\lambda}'}{\partial\lambda^2}\end{bmatrix} \quad (23)$$

where

$$\frac{\partial^2\lambda'}{\partial\lambda^2}=\frac{\mu^2}{V\log(1+\mu)}\;\frac{-1}{(1+\mu/V\text{ abs }(\hat{x}))^2}\;\text{sgn}(\hat{\lambda}) \quad (24)$$

$$\text{sgn}(\hat{\lambda})=1 \text{ if } \hat{\lambda} \text{ is positive}$$

$$=-1 \text{ if } \hat{\lambda} \text{ is negative}$$

$$=0 \text{ if } \hat{\lambda}=0 \quad (25)$$

and $\hat{\lambda}$ has argument $(t|t_k)$.

The type of nonlinear transformation described by equations (24) and (25) is sometimes used in PCM compandors [18]. In our case, this may be considered as a device to limit the peak signal amplitude at the input of the FM modulation so that the frequency deviation does not exceed the frequency bandwidth allotted.

For the present case, the matrix $F(t)$ in equation (4) is given by

$$F=\begin{bmatrix}-a & 0\\ \dfrac{\partial\hat{\lambda}'}{\partial\lambda} & 0\end{bmatrix} \quad (26)$$

where

$$\frac{\partial\hat{\lambda}'}{\partial\lambda}=\frac{\mu}{\log(1+\mu)}\;\frac{1}{1+\mu/V\text{ abs }(\hat{\lambda})}. \quad (27)$$

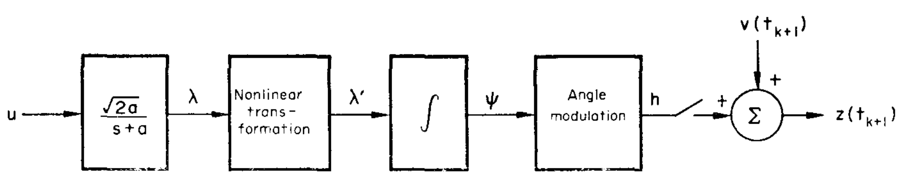

FIG. 3. An example of FM system with nonlinear dynamics.

Clearly, the results of the previous sections can be easily extended to include this case. Some results for a particular sample path of the process $u(\cdot)$ are presented in Figs. 4 and 5 for $\mu=50$ and $V=1$. Note that because of (26), P is now coupled to $\hat{x}$ through F and P_λ fluctuates as time goes on.

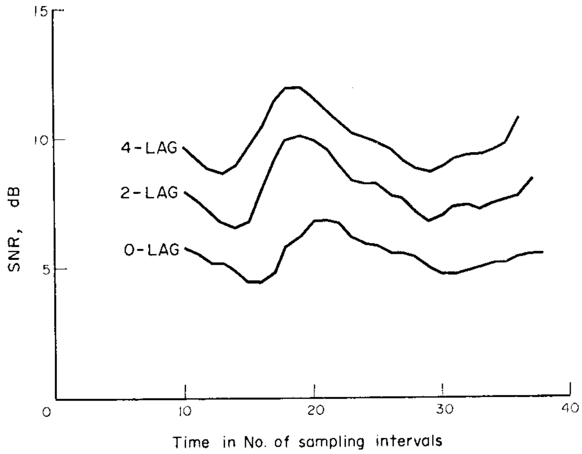

FIG. 4. Performance of demodulator for FM system of Fig. 3 (SNR output for CNR=1200, $\beta=10$, $\theta=10$, $T=0{\cdot}5$).

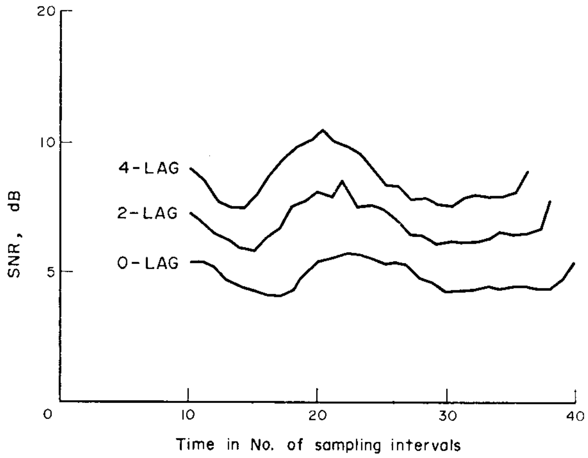

FIG. 5. Performance of demodulator for FM system of Fig. 3 (SNR output for CNR=600, $\beta=10$, $\theta=10$, $T=0{\cdot}5$).

4. *Conclusion*

We have shown that the nonlinears moothing equations in Ref. [11] can be useful for frequency demodulation from discrete-time observations. The performance curve for high carrier-to-noise (CNR) in Figs. 1, 2, 4 and 5 illustrate the improvement obtained using the smoothing equations.

Note that our smoothing equations merely introduce additional elements to the 0-lag demodulator structure. These additional elements do not require additional digital phase-locked loops. In general, if we start with an N-lag demodulator, we can always append an extra block to make an $N+1-$lag demodulator. All the outputs of N, $N-1$, $\ldots$, $0-$lag demodulators are still retained.

The threshold performance of this type of fixed-lag FM demodulators would be the same as the type of FM demodulators described in [4, 14]. This is because both types of demodulation can be thought of being driven by the same approximate innovations process, the accuracy of which determines the threshold.

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Résumé—Cet exposé considère une application de quelques résultats récents d'algorithmes d'adoucissement à délai fixe à la démodulation de mesures de signaux FM bruyants discrets. Le probléme est d'abord attaqué en modelant le système de communications en fonction d'un système stochastique non-linéaire dimensionel fini et en faisant le rapprochement du modèle résultant à un problème d'estimation pour lequel des solutions approximatives peuvent être obtenues. Les résultats de performance sont présentés, utilisant l'algorithme de filtrage modifié, tronqué de second ordre. Les résultats illustrent que si un délai arrive à une estimation optimum on-line, une amélioration notable est possible même avec un délai fixe très modeste.

Zusammenfassung—Betrachtet wird eine Anwendung einiger neuer Ergebnisse von Glättungsalgorithmen mit fester Totzeit auf die Demodulation diskreter verrauschter Messungen von FM-Signalen. Das Problem wird angegriffen, indem zuerst das Kommunikationssystem bezüglich eines nichtlinearen stochastischen Systems endlicher Dimension modelliert wird und das resultierende Modell auf ein Schätzproblem bezogen wird, für das approximierte Lösungen erhalten werden können. Ergebnisse des Verhaltens werden dargestellt, wobei der modifizierte abgebrochene Filteralgorithmus zweiter Ordnung benutzt wird. Die Ergebnisse veranschaulichen, daß, wenn bei der Erreichung einer optimalen "on-line"-Schätzung eine Verzögerung erlaubt ist, eine kennzeichnende Verbesserung sogar bei einer sehr mäßigen festen Totzeit erhalten wird.

Резюме—Эта статья рассматривает некоторые современные результаты применения алгоритма сглаживания с постояннымс запаздыванием для демодуляции дискретных измерений шумов в чм сигналах. Проблема решается с помощью моделирования системы связи в терминах конечно-размерных нелинейных стохастических систем и применения результирующей модели для проблемы оценки для которой может быть получено приближенное решение Приводятся результаты функционирования, при этом используется модифимированный алгоритм фильтрации второго порядка Результаты показывают, что если допустимо запаздывание в получающейся "он лайн" оптимальной оценке замкнутого контура то можно получить существенное улучшение даже при очень умеренном фиксированном запаздывании.