

Stable Realization of Fixed-Lag Smoothing Equations for Continuous-Time Signals

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Abstract—Novel smoother structures are introduced for the optimal fixed-lag smoothing of continuous-time signals in noise. The smoothers have the very desirable property that they are simultaneously realizable, stable in the sense of Lyapunov, and optimal. This is in contrast to those proposed to date which are either optimal and realizable but unstable, realizable and stable but suboptimal, or optimal and stable but unrealizable.

I. INTRODUCTION

In [1] it is pointed out that existing continuous-time fixed-lag smoothers are computationally unstable, as are some proposed discrete-time algorithms.

A number of approaches have been developed, since the appearance of [1], to achieve stable discrete-time fixed-lag smoothing algorithms [2]–[4], but all attempts to extend the various results to continuous-time smoothing have achieved stability only at the expense of either loss of optimality or loss of realizability [4]–[6]. Stated in another way, the performance of proposed suboptimal continuous-time fixed-lag smoothers approaches that of the optimal fixed-lag smoother only as the smoother dimension, delays aside, approaches infinity. Without denying the obvious utility of the results of [5] and [6], it is clear that a novel approach is required to yield continuous-time, fixed-lag smoothers which are simultaneously

realizable, stable, and optimal.

In this note, novel fixed-lag smoother structures are introduced for the optimal fixed-lag smoothing of continuous-time signals in noise. The smoothers are simultaneously realizable (apart from pure time delays), stable,¹ in the sense of Lyapunov, and optimal. The smoothers involve, in loose terms, two optimal “unstable” smoothers back to back, with states of each periodically set to zero to ensure stability and the output switched alternatively from one to the other. The cost of achieving the three desirable properties of realizability, stability, and optimality is thus an increase in complexity on that of the unstable smoother and the introduction of the possible unpleasant feature of discontinuous signals within the smoother.

II. PROBLEM STATEMENT

We have in mind the problem of estimating the state at time t of a lumped linear system of the form

$$\dot{x} = F(t)x + G(t)u \quad (1)$$

where u is Gaussian white noise of mean zero and covariance $Q(t)\delta(t - \tau)$, and the initial state of (1), namely, $x(t_0)$ is a Gaussian random variable, independent of u , zero mean, and having a covariance P_0 . The matrices $Q(t)$ and P_0 are symmetric and nonnegative definite.

The received measurements are

$$z = H'(t)x + v(t) \quad (2)$$

where v is Gaussian white noise, independent of u and $x(t_0)$, of mean zero, and having a covariance $R(t)\delta(t - \tau)$ where $R(t)$ is symmetric positive definite.

Given the measurements z over $[t_0, t]$, the fixed-lag smoothed estimate $\hat{x}(t - \Delta|t) = E[x(t - \Delta)|z(\tau), t_0 \leq \tau \leq t]$ can be expressed as

Manuscript received January 18, 1973; revised September 11, 1973. This work was supported by the Australian Research Grants Committee.
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¹ But not asymptotically stable as in more recent realizations with time-varying gain elements.

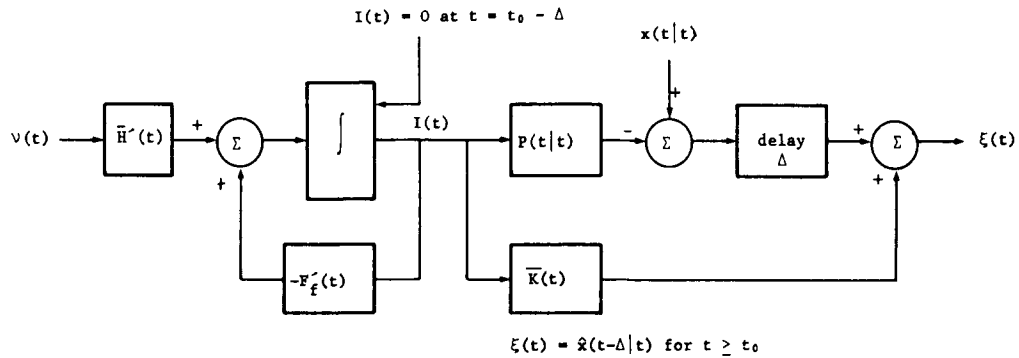


Fig. 1. Unstable fixed-lag smoother.

[8],[9]

$$\hat{x}(t - \Delta|t) = \hat{x}(t - \Delta|t - \Delta) + \bar{K}(t) \int_{t-\Delta}^t \Phi'(\sigma, t) \bar{H}'(\sigma) \nu(\sigma) d\sigma \quad (3)$$

where

$$\bar{K}(t) = P(t - \Delta|t - \Delta) \Phi'(t, t - \Delta) \quad (4)$$

$$\bar{H}'(\sigma) = H'(\sigma) R^{-1}(\sigma) \quad (5)$$

$$\nu(\sigma) = z(t) - H'(t) \hat{x}(t|t) \quad (6)$$

and $\hat{x}(t|t)$, $\Phi(\sigma, t)$, $P(t|t)$ are the filter state, filter transition matrix, and filter error covariance of the now familiar Kalman-Bucy filter [7], given by

$$\dot{\hat{x}}(t) = F_f(t) \hat{x}(t) + K_f(t) z(t), \quad \hat{x}(t_0|t_0) = 0$$

$$F_f(t) = F(t) - K_f(t) H'(t)$$

$$K_f(t) = P(t|t) H(t) R^{-1}(t)$$

$$\dot{P}(t|t) = P(t|t) F'(t) + F(t) P(t|t) - P(t|t) H'(t) R^{-1}(t) H(t) P(t|t) + G(t) Q(t) G'(t)$$

$$P(t_0|t_0) = P_0.$$

Our task is to find stable realizations of (3).

III. REALIZATIONS OF THE FIXED-LAG SMOOTHING EQUATIONS

We first recall the usual, albeit unstable, realization of the fixed-lag smoothing equation. This is obtained from an expanded version of (3) as follows:

$$\begin{aligned} \hat{x}(t - \Delta|t) &= \hat{x}(t - \Delta|t - \Delta) + \bar{K}(t) \int_{t_0}^{t-\Delta} \phi'(\sigma, t) \bar{H}'(\sigma) \nu(\sigma) d\sigma \\ &\quad - P(t - \Delta|t - \Delta) \int_{t_0}^{t-\Delta} \phi'(\sigma, t - \Delta) \bar{H}'(\sigma) \nu(\sigma) d\sigma. \end{aligned} \quad (7)$$

The smoothed estimate $\hat{x}(t - \Delta|t)$ can now be obtained from the output of the time delay dynamical system having state equations derived from (7) as

$$\dot{I}(t) = -F_f'(t) I(t) + \bar{H}'(t) \nu(t), \quad I(t_0 - \Delta) = 0 \quad (8)$$

$$\xi(t) = \bar{K}(t) I(t) - P(t - \Delta|t - \Delta) I(t - \Delta) + \hat{x}(t - \Delta|t - \Delta) \quad (9)$$

where $\hat{x}(t - \Delta|t) = \xi(t)$ for $t \geq t_0$. Notice that the state $I(t)$ is simply the integral term $I(t) = \int_{t_0}^t \phi'(\sigma, t) \bar{H}'(\sigma) \nu(\sigma) d\sigma$ from (7). This dynamical system, known as a fixed-lag smoother (see also Fig. 1) is one of the conventional realizations of the fixed-lag smoothing

equations. Observe that this realization is in fact unstable in the sense of Lyapunov when the optimal filter is stable in the sense of Lyapunov (the usual case) even though the smoothing equation (7) represents a bounded-input bounded-output mapping. All conventional realizations of (3) result in unstable systems [1].

In deriving a stable realization of the smoothing equation (3), the first step is to expand the equation (3) as follows:

$$\begin{aligned} \hat{x}(t - \Delta|t) &= \hat{x}(t - \Delta|t - \Delta) + \bar{K}(t) \int_{\alpha(t)}^t \phi'(\sigma, t) \bar{H}'(\sigma) \nu(\sigma) d\sigma \\ &\quad - P(t - \Delta|t - \Delta) \int_{\alpha(t)}^{t-\Delta} \phi'(\sigma, t - \Delta) \bar{H}'(\sigma) \nu(\sigma) d\sigma \end{aligned} \quad (10)$$

where $\alpha(t)$ is a discontinuous (staircase) function of t defined from

$$\begin{aligned} \alpha(t) &= t_0 + 2(k-1)\Delta, \quad t_0 + (2k)\Delta \leq t < t_0 + (2k+1)\Delta \\ &= t_0 + 2k\Delta, \quad t_0 + (2k+1)\Delta \leq t < t_0 + (2k+2)\Delta \end{aligned} \quad (11)$$

for $k = 0, 1, 2, \dots$ (note that $(t - \Delta) \geq \alpha(t)$ for all t). These equations appear complicated because of the lower limit of the integrations depending in a discontinuous manner on the value of t . However, it is not too difficult to see that an appropriate duplication of the smoother of Fig. 1 with appropriate switching and resetting to zero of the integrators, as described in the Introduction of this note and indicated in Fig. 2(a), is in fact a realization of these equations. Observe that the periodic resetting to zero of the integrators ensures Lyapunov stability of the smoother (but not asymptotic stability of the smoother). Actually, not all the duplications indicated in Fig. 2(a) are necessary, and an obvious simplification yields the realization of Fig. 2(b).

IV. DISCUSSION

The system in Fig. 2(b) requires n scalar delay elements where n is the dimension of the state $x(t)$. These delay elements are required to transmit discontinuous signals due to the periodic resettings of the integrators and the periodic switchovers from one set of integrators to the other. Besides, the system requires both the filtered estimates and the innovation process as inputs. An alternative fixed-lag smoother, derived in [10] and depicted in Fig. 3, where $K_{1a}(t) = P^{-1}(t)G(t)$ and $K_{1b}(t) = Q(t)G'(t)P^{-1}(t)$ requires only the filtered estimates as inputs, avoids the need to delay discontinuous signals, and reduces the number of delay elements required for the case when the dimension of the plant input is less than the state dimension.

When the delay is approximated by a finite dimensional network, the optimal smoother becomes a suboptimal smoother [10]. Such a suboptimal smoother is certainly easier to design than the suboptimal smoother reported in [6]. However, the requirement for switching and for the duplication of components makes such suboptimal

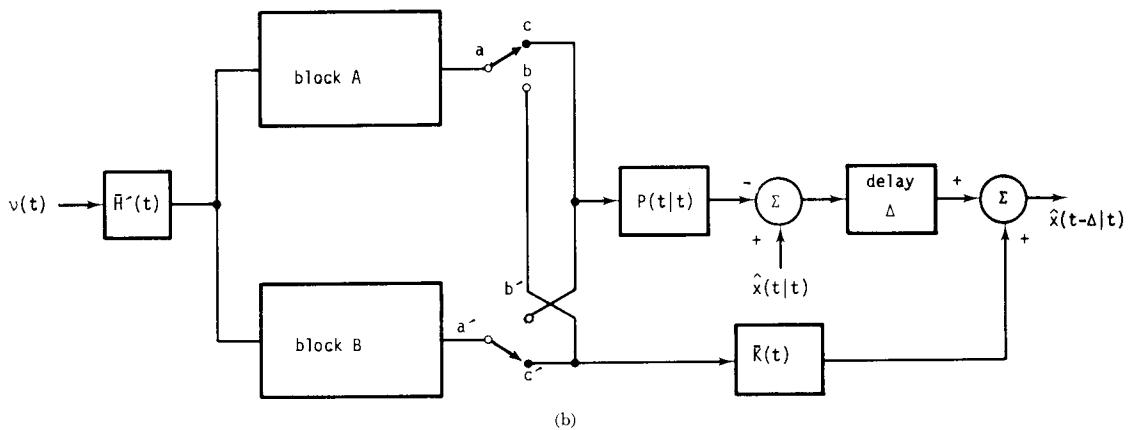
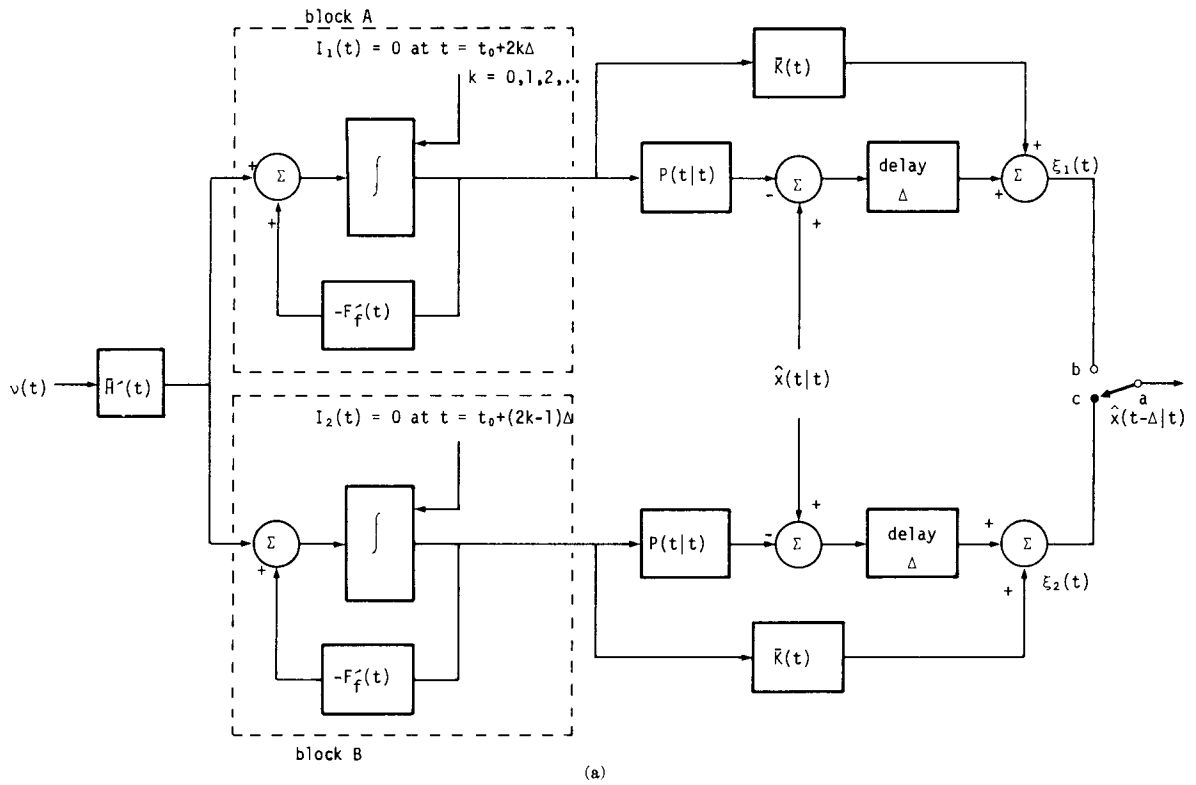


Fig. 2. (a) $\xi_1(t)$ gives $\hat{x}(t|t + \Delta)$ during $[t_0 + 2k\Delta, t_0 + (2k + 1)\Delta]$. $\xi_2(t)$ gives $\hat{x}(t|t + \Delta)$ during $[t_0 + (2k + 1)\Delta, t_0 + (2k + 2)\Delta]$. a b closed at $t = t_0 + (2k - 1)\Delta$. $a'c'$ closed at $t = t_0 + 2k\Delta$. (b) $ab, a'b'$ closed at $t = t_0 + (2k - 1)\Delta$. $ac, a'c'$ closed at $t = t_0 + 2k\Delta$.

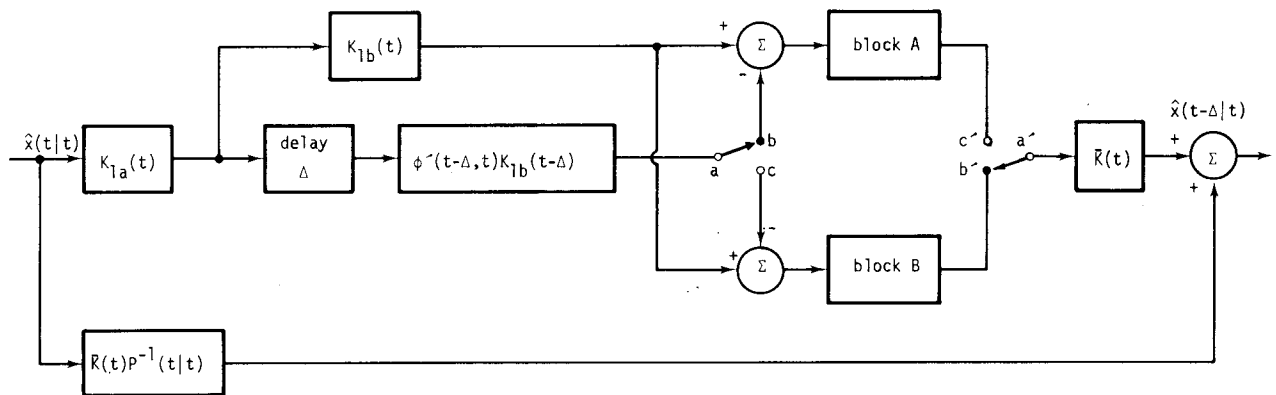


Fig. 3. Fixed-lag smoother using a minimal number of delays. $ab, a'b'$ closed at $t = t_0 + (2k - 1)\Delta$. $ac, a'c'$ closed at $t = t_0 + 2k\Delta$.

smoothers more tedious to implement than the smoothers of [6]. Because of the switchings, the suboptimal smoothers behave as time-varying systems even when the system model is time-invariant and the Kalman filter has reached steady states. It turns out that the performance of the suboptimal smoothers fluctuates periodically with period Δ .

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