A Circle Criterion Generalization for Relative Stability

Abstract—It is shown that the *circle criterion* for linear systems containing a single time-varying element corresponds to a specialization of the Popov stability theory. The circle criteria are generalized using the Popov theory to give some measure of the degree of the system stability.

This correspondence considers a generalization of the well-known *circle criteria* to give the degree of stability of a linear system containing a single time-varying element. The block diagram of the system S under consideration is shown in Fig. 1. We will assume the following:

(S1) $g(s)=q(s)/p(s)$ with $q(s)$ and $p(s)$ being finite polynomials without common factors, $p(s)$ is monic with ρ zeros in the half-plane $\mathrm{Re}\,[s]>0$, and the degree of $p(s)$ exceeds that of $q(s)$.

(S2) $k(t, y)$ is bounded on $[0, \infty)$, and it is smooth enough to guarantee the existence of a solution to the governing differential equations.

The circle criteria[1]–[4] are a generalization of the classical Nyquist stability criteria, useful for predicting the stability of S when the function $k(t, y)$ satisfies a gain limitation of the following form:

(S3) $\alpha \leq k(t, y) \leq \beta$, where α and β are positive constants.

The circle criteria involve an open *critical disk* $D(\alpha, \beta)$ in the g plane, centered at the point $-(\alpha+\beta)/2\alpha\beta$, and having radius $(\beta-\alpha)/2\alpha\beta$ (see Fig. 2). The disk shrinks to the *critical point* of the Nyquist criterion as α and β approach each other. A statement of the circle criteria follows.

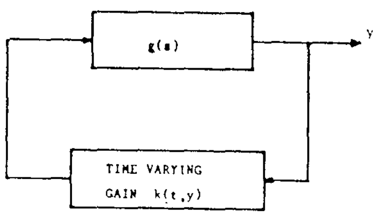

Fig. 1. System S.

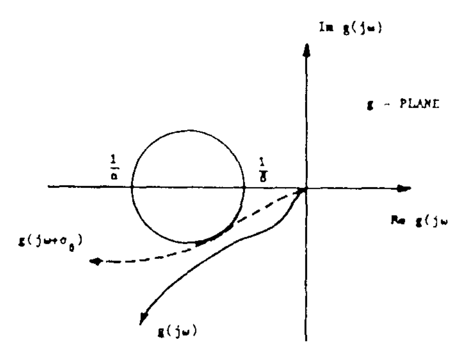

Fig. 2. Nyquist loci and disk $D(\alpha, \beta)$.

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Theorem 1 (Circle Criteria)

For the system S of Fig. 1 with (S1), (S2), and (S3) satisfied, if *either*

(C1) the Nyquist locus $g(j\omega)$ $[g(j\omega+\sigma_0)]$ does not intersect the disk $D(\alpha, \beta)$ and encircles it ρ times in the counterclockwise direction [for some $\sigma_0<0$]

or

(C2), and (C2) implies (C1),

$$\frac{1/\beta+g(s)}{1/\alpha+g(s)}, \left[\frac{1/\beta+g(s+\sigma_0)}{1/\alpha+g(s+\sigma_0)}\right]$$

is positive real [for some $\sigma_0<0$],

then S is stable in the sense that all initial conditions lead to outputs y that are bounded [approach zero] as $t\to\infty$.

The circle criteria are in fact a specialization of some Popov stability theory results in disguise. We first show this to be the case and then apply the Popov theory to give a generalization of the circle criteria.

As a first observation, we note that without loss of generality, α may be taken as zero in Theorem 1. For this case (S3) and (C2) are replaced, respectively, by

(S4) $0\leq k(y, t)\leq\beta$

(C3) $(1/\beta+g(s))$ $[1/\beta+g(s+\sigma_0)]$ is positive real [for some $\sigma_0<0$].

For simplicity, the condition (C1) for the case $\alpha=0$ is not considered in this correspondence.

A statement of the Popov stability theorem[5], [6] and a corollary[1], [7] are now given. For the present development, particular attention should be given in the theorem to the case when σ_0 is an arbitrarily small negative constant and $\gamma=0$.

Theorem 2 (Popov Criterion)

For the system S of Fig. 1, with (S1) and (S2) satisfied, but with the nonlinearity time-invariant and satisfying

(S5) $0\leq k(y)\leq\beta$,

if

(C3) $(1/\beta+(1+\gamma s)g(s))$ $[1/\beta+(1+\gamma s)g(s+\sigma_0)]$ is positive real for some γ satisfying $0\leq\gamma\leq 1$ [and some $\sigma_0<0$],

then the system S is stable in the sense that all sets of initial conditions lead to outputs y that are bounded [approach zero] as $t\to\infty$.

Moreover, a Liapunov function V exists such that

$$\frac{\dot V(x)}{V(x)} \leq \frac{2\sigma_0[x'Px+\gamma k(y)y]}{x'Px+2\gamma\int_0^y k(\rho)d\rho}$$

where P is a positive constant dependent on $g(s)$.

Corollary[5], [9]

For the case $\gamma=0$, the nonlinearity may be time-varying, i.e., (S5) may be replaced by (S4).

We now observe directly that Theorem 1 (circle criterion), stated without loss of generality for the case $\alpha=0$ (see (S4) and (C3)), is in fact a specialization (i.e., the case $\gamma=0$ and σ_0 an arbitrarily small negative constant) of the Popov theory given in Theorem 2 and the Corollary. Theorem 2 is now restated for this case.

Theorem 2A (Special Case of the Popov Criterion)

For the system S of Fig. 1, if (S1), (S2), (S4), and (C3) are satisfied, then the system S is stable in the sense that all sets of initial conditions lead to outputs y that are bounded [approach zero] as $t\to\infty$. Moreover, a Liapunov function V exists such that $\dot V/V<2\sigma_0$.

The circle criteria may now be extended to consider the degree of stability of system S simply by restating all the results of Theorem 2A in terms which may be interpreted on the g plane.

Theorem 3 (Circle Criteria Generalization)

For the system S of Fig. 1, with (S1), (S2), and (S3) satisfied, if

(C4) the Nyquist locus $g(j\omega)$ $[g(j\omega+\sigma_0)]$ does not intersect the disk $D(\alpha, \beta)$ and encircles it ρ times in the counterclockwise direction [for some $\sigma_0<0$] (see Fig. 2),

then S is stable in the sense that all initial conditions lead to outputs that are bounded [approach zero] as $t\to\infty$.

Moreover, a Liapunov function V exists such that $\dot V/V<2\sigma_0$.

Using the link between the Popov theory for nonlinear systems and the circle criteria for time-varying systems, a companion report[9] is in preparation to consider further generalizations and extensions of circle criteria results.

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