Structural Stability of Linear Time-Varying Systems

Abstract—**An application of Popov criterion generalizations for time-varying systems is considered in regard to the tolerance of small amounts of memoryless sector nonlinearities existent in any practical realization of a linear system. It is shown that such nonlinearities can be tolerated if they are sufficiently small, without disturbing system stability.**

Any physical realization of a linear system can never be completely linear, and consequently engineers have had to adopt as an article of faith the assumption that such physical realizations are structurally stable, i.e., that the perturbation of system parameters does not affect the gross performance of the system, particularly such features as the stability of the system.

In this correspondence the structural stability of classes of linear, time-varying systems is considered. To fix ideas more definitely, variations from linear behavior are considered so that a normally linear element is assumed to be composed primarily of a linear (possibly time-varying) part, together with a small amount of time-varying nonlinearity assumed to be confined to a sector. Fig. 1 illustrates the relation between input and output of the element at some fixed value of time t_1.

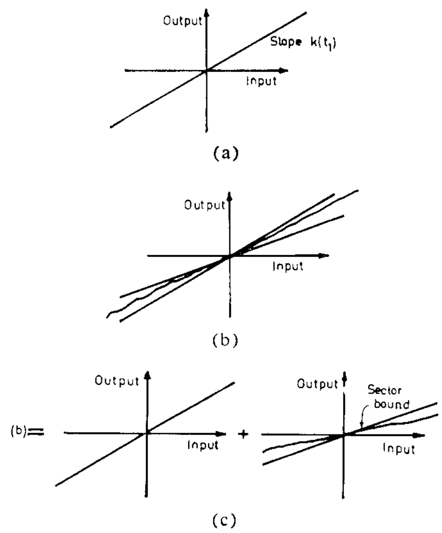

Fig. 1. Nonlinearity characteristics. (a) Ideal. (b) Actual. (c) Decomposition.

For convenience, a single such element is considered. Extension to the case of many such elements is readily possible, and such extension has been done for time-invariant systems.[1] The nonlinear part of the nonideal element is extracted from the remainder of the system to appear in a feedback loop as shown in Fig. 2.

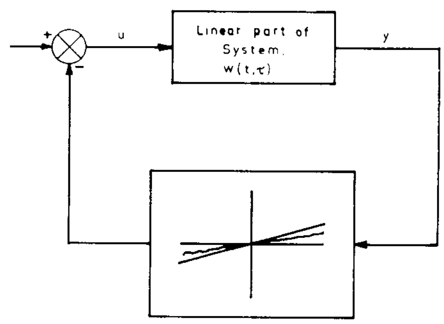

Fig. 2. Closed-loop system.

When the nonlinear sector-limited feedback is absent, stability of the linear system is assumed to prevail. (The precise sort of stability will be discussed subsequently.) The effect of the nonlinear feedback on the system stability must be explored.

Two distinct viewpoints will be taken of the linear part of the closed-loop system, corresponding to both the finite and infinite dimensionality of this component. In the first instance, it is assumed that the input u, the output y, and the state x are related by

$$\dot{x} = Fx + Gu \tag{1a}$$

$$y = H'x \tag{1b}$$

where it is also assumed that F, G, and H are bounded; $[F, G]$ is uniformly completely controllable, $[F, H']$ is uniformly completely observable, and the transition matrix $\Phi(t, \tau)$ associated with (1a) is exponentially bounded, i.e.,

$$\|\Phi(t, \tau)\| < \alpha_1 \exp[-\alpha_2(t - \tau)] \tag{2}$$

for some positive constants α_1 and α_2.

In the second instance, it will be assumed that the impulse response $w(t, \tau)$ is exponentially bounded, i.e.,

$$|w(t, \tau)| < \alpha_3 \exp[-\alpha_4(t - \tau)] \tag{3}$$

for some positive constants α_3 and α_4. (Note that for finite dimensional systems of the sort considered, (3) is a consequence of (1), (2), and the boundedness of G and H.) For infinite dimensional systems, it is additionally required that for the open-loop system any zero input response $y_0(\cdot)$ commencing at an arbitrary time t_0 should be bounded and square integrable.

For the first class of systems, Liapunov stability of the closed-loop system will be considered, while for the second class, consideration is given to the square integrability and boundedness of the output of the closed-loop system. By way of simplification, if either of these conditions prevail, the closed-loop system will be called stable.

Let us now restrict the nonlinearity of Fig. 2 to being confined within a sector bounded by the horizontal axis and a line of slope k^{-1}. Otherwise the nonlinearity is arbitrary. A specialization of results in Moore and Anderson[2] and Moore[3] establishes that the closed-loop system is stable if

$$R(t, \tau) - \eta\delta(t - \tau) = (k - \eta)\delta(t - \tau) + w(t, \tau)1(t - \tau) + w(\tau, t)1(\tau - t) \tag{4}$$

is a covariance for an arbitrary positive constant η. Here $\delta(t-\tau)$ is the unit impulse and $1(t-\tau)$ is the unit step function. In terms of the notation used in Moore and Anderson[2] and Moore[3] the matrices $A(\cdot)$ and $B(\cdot)$ of these references are taken as the identity and zero matrices, respectively.

Our goal here is to show that with (3) holding (recall that this is assumed for infinite dimensional systems, and is a consequence of the assumptions made for finite dimensional systems), there exists a k for which the quantity $R(t, \tau) - \eta\delta(t - \tau)$ in (4) is a covariance. In other words, it is *always* possible to tolerate some sector nonlinearity without perturbing stability.

The key to demonstrating the existence of a k is to note that $w(t, \tau)1(t - \tau)$ maps L_2 functions into L_2 functions and, as an operator, is bounded. To see this, consider the inequalities

$$|y(t)| = \left|\int_{t_0}^{t} w(t, \tau)u(\tau)d\tau\right|$$
$$\leq \int_{t_0}^{t} |w(t, \tau)|\,|u(\tau)|\,d\tau$$
$$\leq \int_{t_0}^{t} \alpha_3 \exp[-\alpha_4(t-\tau)]\,|u(\tau)|\,d\tau. \tag{5}$$

Now it is known (see Titchmarsh,[4] theorem 65) that $\exp[-\alpha_4(t-\tau)]1(t-\tau)$ for positive α_4 is a bounded map of L_2 functions into L_2 functions; moreover $|u(\cdot)|$ is an L_2 function if $u(\cdot)$ is an L_2 function. Consequently $|y(\cdot)|$ is majorized by an L_2 function and is thus an L_2 function itself. Moreover a bound of $w(t, \tau)1(t-\tau)$ evidently exists.

The adjoint of the operator $w(t, \tau)1(t-\tau)$ is $w(\tau, t)1(\tau-t)$ and must map L_2 into L_2; in fact, it has the same norm as $w(t, \tau)1(t-\tau)$.

Now consider the integral

$$I = \int_{T_1}^{T_2}\int_{T_1}^{T_2} [R(t, \tau) - \eta\delta(t - \tau)]u(t)u(\tau)dtd\tau \tag{6}$$

which must be positive for every T_1, T_2, and $u(\cdot)$ for (4) to define a covariance. Setting a bound on the operator $w(t, \tau)1(t-\tau)$ of W, it follows that

$$I \geq \int_{T_1}^{T_2} (k - \eta)u^2(t)dt - 2W\int_{T_1}^{T_2} u^2(t)dt. \tag{7}$$

Consequently, by choosing k and η such that $k - \eta > 2W$, the positivity of I may be guaranteed. In other words, by taking the slope k^{-1} of the nonlinearity bound small enough (in fact, smaller than a quantity determined from $w(t, \tau)$) the closed-loop system is guaranteed stable.

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