Optimum Differentiation Using Kalman Filter Theory

Abstract—**Consideration is given to the construction of an optimum differentiator to give the minimum-variance unbiased estimate of the first derivatives of random signals corrupted by white noise. *It is assumed that the* signals are differentiable and are the outputs of a known linear finite-dimensional (possibly time-varying) system excited by white noise. Extension of the results to consider higher-order differentiation is straightforward.**

This letter presents an application of Kalman state estimator theory[1] to the construction of an optimum differentiator giving the minimum-variance unbiased estimate of the first derivatives of random signals corrupted by white noise. It is assumed that the signals are differentiable and are the outputs of a known linear finite-dimensional system excited by white noise. The results constitute an indirect extension of the Wiener optimum differentiator[2] given for the case when the signal spectral density is known. For the case when the signal covariance is specified a more direct extension of the Wiener results is possible using the results of this letter together with the results of another work.[3]

The Kalman state estimator[1] gives the best minimum-variance unbiased estimate of the states of a linear dynamical finite-dimensional (possibly time-varying) system excited by white noise u when measurements $z(\cdot)$ of the system output $y(\cdot)$ corrupted by white noise $w(\cdot)$ are available over a time interval $[t_0, t_1]$. Thus

$$z = y + w \tag{1}$$

where z, y, and w are all m-vectors. The white noise u and w have zero mean and covariances

$$\text{cov}\left[u(t), u(\tau)\right] = I_m\delta(t-\tau) \tag{2}$$

$$\text{cov}\left[w(t), w(\tau)\right] = R\delta(t-\tau) \tag{3}$$

where R, possibly time-varying, is positive definite symmetric for all time t, and $\delta(t)$ is the delta function at time t. The noises $u(\cdot)$ and $w(\cdot)$ are independent. Consider that the signal y is the output of the system having the state space equations

$$\dot{x} = Fx + Gu; \qquad y = H'x \tag{4}$$

where x is an n-vector and is the system state, and F, G, and H' are known $n\times n$, $n\times m$, and $m\times n$ matrices, respectively. It is assumed that the covariance of the states of (4) at time t_0, i.e., $\text{cov}\,[x(t_0), x(t_0)]$, is known.

Denoting the minimum-variance unbiased estimate of x by $\hat{x}$, the state space equations of the state estimator are then given as

$$\dot{\hat{x}} = (F - KH')\hat{x} + Kz \tag{5}$$

where

$$K = PHR^{-1} \tag{6}$$

and P is given as the solution of the matrix Riccati differential equation

$$\dot{P} = PF' + FP - PHR^{-1}H'P + GG' \tag{7a}$$

$$P(t_0) = \text{cov}\left[x(t_0), x(t_0)\right]. \tag{7b}$$

We now assume that $\dot{H}$ exists. Then by using (4), $\dot{y}$ can be calculated as

$$\begin{aligned}\dot{y} &= \dot{H}'x + H'\dot{x}\\ &= (\dot{H}' + H'F)x + H'Gu.\end{aligned} \tag{8}$$

This means that for the case when H is differentiable and

$$H'G = 0 \tag{9}$$

the derivative $\dot{y}$ is given by

$$\dot{y} = (\dot{H}' + H'F)x \tag{10}$$

and therefore the minimum-variance unbiased estimate of $\dot{y}$, written $\hat{\dot{y}}$, is given as

$$\hat{\dot{y}} = (\dot{H}' + H'F)\hat{x}. \tag{11}$$

We conclude that for the case *when* (9) *is satisfied and H is differentiable, the optimum differentiator has the state space equations*

$$\dot{\hat{x}} = (F - KH')\hat{x} + Kz \tag{5}$$

$$\hat{\dot{y}} = (\dot{H}' + H'F)\hat{x}. \tag{11}$$

Note: For the case when (9) is not satisfied, i.e., for the case

$$H'G \neq 0, \tag{12}$$

the signal derivative $\dot{y}$ [see (8)] will contain a component of the input white noise. To estimate this component and to incorporate the results into the differentiator would have no significance for practical situations.

The results are readily extended to cover the cases when infinite time intervals are considered and for the cases when higher derivatives of the signals are to be estimated.

As might be expected, the condition (9) is necessary and sufficient for the covariance of $\dot{y}$ to contain no delta function. Note that the resulting continuity of $\text{cov}[\dot{y}(t), \dot{y}(\tau)]$ at every point on $t=\tau$ means that the stochastic process $y(t)$ is differentiable in the mean square sense.[2]

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References

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Ultrasonic Transducer Configuration for Producing a Phase Grating of Nearly Uniform Strength

Abstract—**A configuration is given which produces nearly uniform phase retardation of a light beam passing through the near field of an ultrasonic transducer.**

As light propagates through a sound field, its phase is retarded proportionally to the local sound pressure. In the Raman-Nath limit[1] of narrow sound fields of low frequency, the total retardation in optical path length may be calculated from the following integral:

$$\theta(x, z) = K\int_{-\infty}^{\infty} p(x, y, z)\,dy$$

where it is assumed that the light is travelling in the y-direction, the sound pressure $p(x, y, z)$ is non-negligible only in a small region of y, and K is the piezooptic coefficient. The magnitude of the quantity θ when multiplied by $2\pi/\lambda$ (λ being the wavelength of the light) is called the Raman-Nath parameter $v(x, z)$ and is a measure of the strength of the effective optical phase grating. In most considerations, the variations of $v(x, z)$ with x and z are either ignored, minimized by restricting the size of the light beam, or averaged in some very nonlinear manner. As the light beam is usually passed through the so-called near field of the transducer, the variations of $v(x, z)$ are actually found to be quite large. This in effect gives an optical grating of locally varying strength which may degrade the effectiveness of a device sampling a large region of the sound field or needing a wide range of frequencies. In this letter, we wish to demonstrate

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