Reprinted by permission from IEEE TRANSACTIONS ON AUTOMATIC CONTROL
Vol. AC-14, No. 1, February 1969
pp. 108-109

Two-Dimensional "Circle Criterion" on the Parameter Plane

Abstract—A graphical procedure using parameter plane theory is given for studying the stability of systems consisting of a linear time-invariant subsystem with two feedback time-varying gains.

In a previous paper [1], a graphical interpretation is given of the well-known "circle criterion" [2], [3] on the parameter plane for the case when one nonlinearity is involved. The advantage of the parameter plane approach for this case is that the influence of system parameters other than loop gain may be considered in a straightforward manner. However, for the case when only the effects of variation of the loop gain bounds are of interest, the standard graphical interpretation on the complex plane is the most straightforward.

This correspondence gives a parameter plane interpretation of the circle criterion for the case when two time-varying gains are involved [3] and the effect of any one of the system parameters is to be determined. For this case, a complex plane approach is not evident.

The two-dimensional circle criterion is now stated as a special case of the multidimensional criterion in [3].

Stability Criterion: Consider a system $\boldsymbol{S}$ consisting of a linear time-invariant finite-dimensional subsystem $\boldsymbol{W}$ with feedback time-varying gains $\boldsymbol{K}$. Consider also that the 2×2 transfer function matrix $W(s)$ of $\boldsymbol{W}$ has the property that $W(\infty) = 0$, and that the bounds of the gains $\boldsymbol{K}$ are $-K_2$ and $-K_1$ with K_1 and K_2 diagonal 2×2 matrices and $(K_1 - K_2)$ nonnegative definite. Then if a diagonal positive definite 2×2 matrix A can be chosen such that

$$Z(s) = A(K_1 - K_2)^{-1} + AW(s)[I + K_2W(s)]^{-1} \quad (1)$$

is positive real, then the system is stable in the sense of Liapunov.

For the case when $W(s)[I + K_2W(s)]^{-1}$ is asymptotically stable, the requirement that $Z(s)$, given by (1), be positive real reduces to simply requiring that $[Z(j\omega) + Z^T(-j\omega)]$ be nonnegative definite for all real ω. For a 2×2 matrix this is equivalent to requiring that the diagonal elements and the determinant of $[Z(j\omega) + Z^T(-j\omega)]$ be nonnegative for all ω.

For the case when $Z(\cdot)$ is a function of two parameters α and β (as well as of s), nonnegativity of the various terms may be checked on an $\alpha\beta$ plane diagram [1]. An example is now considered to illustrate the application of the theory.

Manuscript received September 9, 1968. This work was supported in part by the Australian Research Grants Committee.

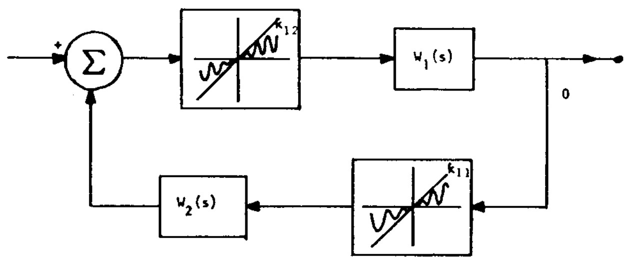

Fig. 1. System block diagram.

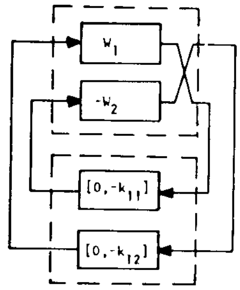

Fig. 2. Parameter-plane diagram.

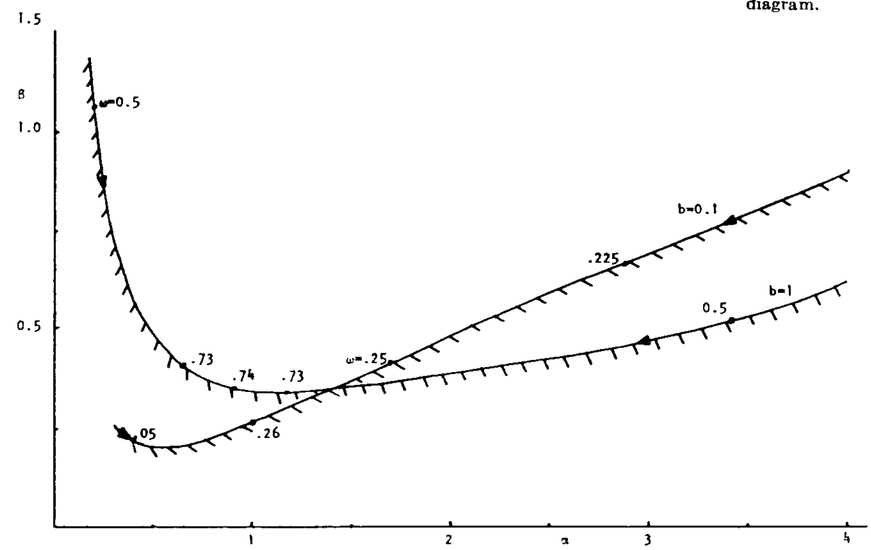

Fig. 3.

Example: Consider the single-loop system of Fig. 1 consisting of stable transfer functions $W_1(s)$ and $W_2(s)$ and time-varying gains bounded by $[0, k_{11}]$ and $[0, k_{12}]$ as indicated. This system rearranged to be in the standard form for the application of the multidimensional circle criterion is given in Fig. 2.

The matrices $W(s)$, K_1, and K_2 of the circle criterion statement are readily identified as

$$W(s) = \begin{bmatrix} 0 & W_1(s) \\ -W_2(s) & 0 \end{bmatrix}$$
$$K_1 = \begin{bmatrix} k_{11} & 0 \\ 0 & k_{12} \end{bmatrix}, \quad K_2 = 0 \quad (2)$$

and without loss of generality an A matrix may be chosen as $\mathrm{diag}\{a, a^{-1}\}$, where a is a positive constant. This means that $[Z(s) + Z^T(-s)]$ may be written as

$$Z(s) + Z^T(-s) = \begin{bmatrix} 2k_{11}^{-1}a & aW_1(s) - a^{-1}W_2(-s) \\ aW_1(-s) - a^{-1}W_2(s) & 2k_{22}^{-1}a^{-1} \end{bmatrix}. \quad (3)$$

Since $W_1(s)$ and $W_2(s)$ are asymptotically stable, for this example the condition that $Z(s) + Z^T(-s)$ be positive real [see (1)] reduces to simply requiring that $D = \det [Z(j\omega) + Z^T(-j\omega)] \geq 0$ for all real ω. That is, we require that $R(a^2,\omega) \geq 0$ for some a^2 and all real ω, where

$$a^2D(a^2,\omega) = 4(k_1k_2)^{-1}a^2 - [a^2W_1(-j\omega) - W_2(j\omega)] \cdot [a^2W_1(j\omega) - W_2(-j\omega)]. \quad (4)$$

This criterion may be considered on a parameter plane diagram as the following indicates.

Using the notation $R_1(\omega)$ and $I_1(\omega)$ $[R_2(\omega)$ and $I_2(\omega)]$ to denote the real and imaginary parts of $W_1(j\omega)[W_2(j\omega)]$ and identifying the parameter plane coordinates α and β as

$$\alpha = a^2 \quad \beta = (k_1k_2)^{-1} \quad (5)$$

the expression for $\alpha D(a^2,\omega)$ may be written as

$$\alpha D(\alpha,\beta,\omega) = 4\alpha\beta - \{[\alpha R_1(\omega) - R_2(\omega)]^2 + [\alpha I_1(\omega) + I_2(\omega)]^2\}. \quad (6)$$

Differentiating the preceding with respect to ω and using primes to denote this differentiation gives

$$\begin{aligned}\alpha D'(\alpha,\beta,\omega) = &-2[\alpha R_1(\omega) - R_2(\omega)] \\ &\cdot[\alpha R_1'(\omega) - R_2'(\omega)] \\ &- 2[\alpha I_1(\omega) + I_2(\omega)] \\ &\cdot[\alpha I_1'(\omega) + I_2'(\omega)]. \quad (7)\end{aligned}$$

The envelope in the $\alpha\beta$ plane of the contours $\alpha D(\alpha,\beta,\omega) = 0$ with ω as parameter is given as the solution of

$$\alpha D(\alpha,\beta,\omega) = 0, \quad \alpha D'(\alpha,\beta,\omega) = 0 \quad (8)$$

for $\omega \in [0, \infty)$. These curves may be shaded according to the sign of the Jacobian

$$J\begin{pmatrix}\alpha D & \alpha D' \\ \alpha & \beta\end{pmatrix}$$

and the region in the $\alpha\beta$ plane for which $\alpha D \geq 0$ is thereby indicated (see [1] for the full theory). Thus it is possible to determine the minimum β (maximum $k_{11}k_{12}$) for which $\alpha D \geq 0$ or, equivalently, for which the system of Fig. is stable.

In Fig. 3, the envelope curves are plotted and shaded when $W_1(s)$ and $W_2(s)$ are

$$W_1(s) = \frac{1}{(s+1)^2}, \quad W_2(s) = \frac{b}{(s+b)} \quad (9)$$

and b is a parameter which is varied. From Fig. 3 for the case $b = 1$, for example, the maximum loop gain $k_{11}k_{12}$ for which stability is guaranteed is $k_{11}k_{12} = 2.9$. It is interesting to compare this result with that obtained using the one-dimensional circle criterion when the two time-varying gains are lumped together. For this case the maximum $(k_{11}k_{12})$ for which stability is guaranteed is $(k_{11}k_{12}) = 4.0$. Again, a comparison may be made for the case when the gains k_{11} and k_{12} are time invariant; for this case max $(k_{11}k_{12}) = 8.0$.

J. B. Moore
A. I. Bartlett
Dept. of Elec. Engrg.
University of Newcastle
Newcastle, N.S.W., Australia

References

[1] J. B. Moore, "Circle criteria in the parameter plane," *Proc. IEE* (London), vol. 115, pp. 557–580, April 1968.

[2] I. W. Sandberg, "A frequency domain condition for the stability of systems containing a single time-varying element," *Bell Sys. Tech. J.*, vol. 43, pp. 1601–1638, July 1964.

[3] J. B. Moore and B. D. O. Anderson, "Generalization of the circle criterion," Preprints, 1968 JACC (Ann Arbor, Mich.), pp. 812–819.

[4] J. B. Moore, "A circle criterion generalization for relative stability," *IEEE Trans. Automatic Control (Correspondence)*, vol. AC-13, pp. 127–128, February 1968.

[5] J. B. Moore and B. D. O. Anderson, "A generalization of the Popov criterion," *J. Franklin Inst.* (to be published).