Lyapunov Function Construction for a Class of Discrete Time-Varying Systems

Abstract—The construction of Lyapunov functions for discrete systems which may be arranged in the form of a linear, time-varying subsystem with feedback memoryless nonlinearities is considered. The results constitute a generalization of Popov's stability theory.

This correspondence considers the application of a recent optimal control theory result [1] to give a method of constructing Lyapunov functions for discrete systems which may be arranged in the form of a linear time-varying subsystem with feedback memoryless nonlinearities. The corresponding continuous-time problem has been considered in [2]. Previous stability results for discrete systems have been restricted to the case when the linear subsystem is time invariant (see [3], [4]).

Consider the system having state equations

$$\begin{aligned} x(k+1) &= F(k)x(k) + G(k)\,\phi[y(k)] \\ y(k) &= H'(k)x(k) \end{aligned} \tag{1}$$

where $\{k\}$ is the indexing set, $y(\cdot)$ is the system output, $\phi[y(k)]$ is an nth order diagonal matrix representing the n memoryless nonlinearities, and $F(\cdot)$, $G(\cdot)$, and $H(\cdot)$ are the system matrices. Consider now as a tentative Lyapunov function the function

$$V(k) = x'(k)\bar{P}(k-1)x(k) + 2\int_0^{y(k)} \phi'(s)B(k)\,ds \tag{2}$$

where $\bar{P}(\cdot)$ is the limit

$$\bar{P}(k) = \lim_{\substack{\epsilon\to 0\\ k_f\to\infty}} P(k,k_f,\epsilon). \tag{3}$$

The value $P(\cdot)$ is the solution of the Riccati difference equation

$$\begin{aligned} P(k-1) = {} & \rho^{-2}F'PF + (C-\rho^{-2}F'PG) \\ & \cdot (R+\epsilon I-\rho^{-2}G'PG)^{-1} \\ & \cdot (C-\rho^{-2}F'PG)' \\ & + \mu B[F'H(k+1)-H] \\ & \cdot [F'H(k+1)-H]', \\ & P(k_f) = \epsilon I \end{aligned} \tag{4}$$

where

$$R = 2AK^{-1} + 2BH'(k+1)G - \mu BG'H(k+1)H'(k+1)G \tag{5}$$

$$C = H(A-B) + F'H(k+1)B - \mu B[FH(k+1)-H]H'(k+1)G \tag{6}$$

with μ and ϵ positive constants and $0<\rho<1$. The matrices B, A, and K are nth-order diagonal matrices. Where indexes are omitted the index (k) is understood.

We now examine $\Delta V \triangleq V(k) - V(k+1)$

$$\begin{aligned} \Delta V = {} & x'(k)\bar{P}(k-1)x(k) \\ & - x'(k+1)\bar{P}(k)x(k+1) \\ & + 2\int_0^{y(k)} \phi'(s)B(k)\,ds \\ & - 2\int_0^{y(k+1)} \phi'(s)B(k+1)\,ds. \end{aligned} \tag{7}$$

Substituting (1), (3), and (4), and rearranging the integral terms gives

$$\begin{aligned} \Delta V = {} & (\rho^{-2}-1)x'F'\bar{P}Fx - \phi'G'\bar{P}G\phi \\ & + 2x'F'\bar{P}G\phi + \lim_{\epsilon\to 0} x'(C-\rho^2F'PG) \\ & \cdot (R+\epsilon I-\rho^{-2}G'PG)^{-1} \\ & \cdot (C-\rho^{-2}F'PG)'x + x'\mu B[F'H \\ & \cdot (k+1) - H]\,[F'H(k+1)-H]'x \\ & + 2\int_0^{y(k+1)} \phi'(s)[B(k) \\ & - B(k+1)]\,ds \\ & + 2\int_{y(k+1)}^{y(k)} \phi'(s)B(k)\,ds. \end{aligned} \tag{8}$$

The second integral term may be expanded using the mean value theorem as (see [3])

$$\begin{aligned} 2\int_{y(k+1)}^{y(k)} & \phi'(s)B(k)\,ds \\ & \geq 2B\phi'(k)[y(k)-y(k+1)] \\ & \quad - \mu B[y(k)-y(k+1)]' \\ & \quad \cdot [y(k)-y(k+1)] \end{aligned} \tag{9a}$$

for some positive

$$\mu > \left|\frac{\partial\phi}{\partial y}\right|_{\max} > 0. \tag{9b}$$

Using (1), (3), (5), (6), and (9), the expression (8) for ΔV may be manipulated and expressed as

$$\begin{aligned} \Delta V \geq {} & \lim_{\substack{\epsilon\to 0\\ k_f\to\infty}} [(R-\rho^{-2}G'PG+\epsilon I)^{-1} \\ & \cdot (C-\rho^{-2}F'PG)'x - \phi]' \\ & \cdot (R-\rho^{-2}G'PG+\epsilon I) \\ & \cdot [(R-\rho^{-2}G'PG+\epsilon I)^{-1} \\ & \cdot (C-\rho^{-2}F'PG)x - \phi] \\ & + 2A\phi'(y-k^{-1}\phi) \\ & + 2\int_0^{y(k+1)} \phi'(s) \\ & \cdot [B(k)-\rho^{-2}B(k+1)]\,ds \\ & + (\rho^{-2}-1)V(k+1). \end{aligned} \tag{10}$$

Clearly provided that

a) $\lim_{\substack{\epsilon\to 0\\ k_f\to\infty}} [R+\epsilon I-\rho^{-2}G'PG] > 0$

b) $A(k) \geq 0, \quad B(k) \geq \rho^{-2}B(k+1) > 0$

c) $K(k)y'(k)y(k) \geq y'(k)\phi[y(k)] \geq 0$

for all k, $V(\cdot)$ as defined in (2) will be a Lyapunov function and $V(k+1) \leq \rho^2 V(k)$. That is, the system is exponentially asymptotically stable. Certainly c) will be satisfied for some sector bound $K(\cdot)$ for any memoryless nonlinearity, b) will be satisfied with an appropriate choice of $A(\cdot)$ and $B(\cdot)$, and a) will be satisfied provided that (see [1])

$$\begin{aligned} z(k,l) = {} & 1/2\rho^2R(k)\delta(k-l) \\ & + \rho C'(k)\Phi(k,l+1)G(l)I(k-l) \end{aligned} \tag{11}$$

is a passive impulse response. (By this is meant that $[z(k,l) + z'(l,k)]$ is a covariance function.) Note that $\Phi(\cdot,\cdot)$ is the transition matrix of system (1), $\delta(\cdot)$ is the unit delta function, and $I(\cdot)$ the unit step function.

Now using (5), (6), and (11), $z(t,\tau)$ may be written in terms of the impulse response

$$w(t,\tau) = H'(k)\Phi(k,l+1)G(l)I(k-l) \tag{12}$$

as

$$\begin{aligned} z(k,l) = {} & \rho^2AK^{-1}\delta(k-l) + \rho(A-B) \\ & \cdot w(k,l) + \rho^2Bw(k+1,k) \\ & + \rho Bw(k+1,l)I(k-l) \\ & - \rho\mu Bw'(k+1,k)[w(k+1,l) \\ & - w(k,l) + \rho w(k+1,k)]. \end{aligned} \tag{13}$$

The stability criterion that $z(k,l)$, given by (13), be passive reduces to the well-known Popov criterion for discrete systems (see [3], [4]). It is clear there is no straightforward method of testing this criterion for the preceding general case other than to solve the Riccati equation and check that (9) is satisfied.

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