

Using the empirical equations given by COOK and FUNK [1], the z 's for systems 1, 2 and 3 under the same conditions are found to be 1.1, 0.1 and 0.89 respectively. It is believed that these results are in error.

Conclusion

In this paper was presented an exact solution to the optimal linear regulator problem with control energy constraint. The proposed numerical method for the computation of the optimal control law is simple and direct. It need not be tailored specifically each time in its use for different systems or for different points of the state space of the same system. Hence, this method lends itself with remarkable ease to real-time implementation.

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References

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 [2] R. E. KALMAN: Contribution to the theory of optimal control. *Bol. Soc. Mex.* **5**, 102–119 (1960).

Résumé—L'article présente la solution exacte du problème du régulateur optimal linéaire avec une contrainte de l'énergie de commande. Un schéma itératif pour le calcul de la solution du problème est proposé. L'article comprend des exemples dans le but d'illustration et de comparaison avec les résultats antérieurement publiés.

Zusammenfassung—Für das optimale lineare Regelungsproblem mit Hilfsenergiebeschränkung wird die genaue Lösung angegeben. Für die Berechnung der Lösung wurde ein iteratives Schema vorgeschlagen. Die angeführten Beispiele dienen zur Veranschaulichung und zum Vergleich mit früher veröffentlichten Resultaten.

Резюме—Статья дает точное решение проблемы оптимального линейного регулятора с ограничением энергии управления. Предлагается повторительная схема для вычисления решения проблемы. Статья включает примеры для иллюстрирования и для сравнения с прежде опубликованными результатами.

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A Note on a Singular Optimal Control Problem*

Une note sur un problème singulier de commande optimale

Bemerkungen zu einem singularen optimalen Regelungssystem

Замечания относительно одной особенной проблемы оптимального управления

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Summary—The dual mode optimal control problem to find a control u to minimize

$$J = \frac{1}{2} \int_0^{\infty} x' Q x dt$$

subject to the constraints $Q > 0$, $\dot{x} = Fx + gu$, $|u| \leq 1$, is solved for its "singular" solutions using well known quadratic regulator theory. The significance of the approach, apart from any due to its simplicity, is that it may be readily extended to handle cases when Q is positive semidefinite. It is also quite straightforward to extend the results to the time-varying multiple-input case.

CONSIDER the completely controllable system having state equations

$$\dot{x} = Fx + gu; \quad |u| \leq 1; \quad x(0) = x_0. \quad (1)$$

The singular control problem of interest is to find a feedback law $u(x)$ which minimizes the performance index

$$J = \lim_{T \rightarrow \infty} \frac{1}{2} \int_0^T (x' Q x) dt; \quad Q = Q' \geq 0. \quad (2)$$

For this class of problem the optimal controller provides dual mode control; one mode, the "bang-bang" mode is

when $u(x)$ takes on the values $+1$ or -1 , while the other mode, the "singular" mode is when $|u| < 1$. Optimal control results for this latter mode have been determined for the case when $Q > 0$ and F is in companion matrix form using Pontryagin's Maximum Principle equations and by giving consideration to the eigenvalues of a system defined from the various equations involved [1, 2]. However the method is not, at least to the author's knowledge, readily extendable to time-varying and/or multiple input systems.

This note shows how the above problem may be solved, once again for its optimal "singular" solutions, using well known quadratic minimization results. For simplicity we restrict attention to the case when $Q > 0$ and when F is non-singular. These conditions are relaxed in Ref. [3]. Reference [3] gives a natural extension of the theory of this note to solve the "singular" problem when time-varying and multiple-input systems are involved.

Consider u_1 and x_1 defined from the equations

$$\dot{u}_1 = -u; \quad x_1 = gu_1 + x. \quad (3)$$

Direct substitution of (3) into (1) and (2) yields

$$\dot{x}_1 = Fx_1 + g_1 u_1 \quad (4)$$

$$J = \lim_{T \rightarrow \infty} \frac{1}{2} \int_0^T (x_1' Q x_1 + 2x_1' h u_1 + u_1' r u_1) dt \quad (5)$$

where

$$h = -Qg; \quad r = g'Qg; \quad g_1 = -Fg. \quad (6)$$

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For the case when F is nonsingular, it is clear from the relationship $g_1 = -Fg$ that the system (4) is completely controllable. We may also write using (6) the following:

$$r > 0; \quad (Q - hr^{-1}h') \geq 0. \quad (7)$$

The latter inequality is derived in [3].

Well known quadratic minimization theory [4] may now be applied to give the optimal control u^* for (4) and the minimum index J^* as

$$u_1^* = -k'x_1; \quad J^* = x_1'(0)Px_1(0) \quad (8)$$

respectively, with P and k defined as

$$P = \lim_{T \rightarrow \infty} \Pi(t, T); \quad k' = r^{-1}(g_1'P + h') \quad (9)$$

with $\Pi(\cdot, \cdot)$ the solution of

$$\begin{aligned} -\dot{\Pi} &= (F - g_1r^{-1}h') + (F' - hr^{-1}g_1')\Pi \\ &\quad - \Pi g_1r^{-1}g_1'\Pi - hr^{-1}h' + Q \end{aligned} \quad (10a)$$

$$\Pi(T, T) = 0. \quad (10b)$$

The various conditions imposed ensure that the solution $\Pi(\cdot)$ of (10) exists and also that the limit P exists and is independent of time t .

We now develop some simple relationships which arise directly from the above equations. From (6) and (10) we have that

$$\begin{aligned} -\dot{\Pi}g &= (F' - hr^{-1}g_1' - \Pi g_1r^{-1}g_1')\Pi g; \\ \Pi(t_1, t_1)g &= 0. \end{aligned}$$

Clearly $\Pi g = 0$ and thus from (9) $Pg = 0$. This means that from (3) and (8)

$$\begin{aligned} J^* &= x_1'(0)Px_1(0) = [gu_1(0) + x_0]'P[gu_1(0) + x_0] \\ &= x_0'Px_0. \end{aligned}$$

A further simple relationship comes from (6) and (9) and the result that $Pg \equiv 0$. We have

$$k'g = r^{-1}(g_1'Pg + h'g) = -r^{-1}g'Qg = -1.$$

This result together with (3) and (8) gives that when $u_1 = u_1$

$$k'x = k'x_1 - k'gu_1^* = k'x_1 - (-1)(-k'x_1) = 0.$$

This means that when $u_1 = u_1^*$, in order for (3) to be satisfied we must choose $x(0)$ such that $k'x(0) = 0$. Such a choice of $x(0)$ gives from (3) that one choice of $x_1(0)$ is $x_1(0) = x(0)$. Let us denote u^* as the control corresponding to u_1 . Then

$$\begin{aligned} u^* &= -\dot{u}_1^* = k'\dot{x}_1 = k'(gu_1^* + \dot{x}) \\ &= k'g\dot{u}_1^* + k'Fx + k'gu^* = k'Fx. \end{aligned}$$

We now make the important observation that with $x(0)$ chosen such that $k'x(0) = 0$, the application of the control $u^* = k'Fx$ to system (1) will minimize the performance index (5) subject to the constraints (3) since this is equivalent to applying u_1 to system (4). Now with (3) holding and (5) minimized, clearly the index (2) will be minimized since with the constraints (3) holding it is identical to index (5).

Once again we denote the minimum index [this time the index (2)] by J^* . The fact that J^* is finite implies, [see (2)] that $x'Qx \rightarrow 0$ as $t \rightarrow \infty$. Clearly, $x \rightarrow 0$ as $t \rightarrow \infty$ since $Q > 0$. In other words the closed loop system $\dot{x} = (I - gk')Fx$ is asymptotically stable and in fact, we have $x'Px$ as a Lyapunov function provided that, first, the initial state $x(0)$ is chosen so that $k'x(0) = 0$ and $|k'Fx(0)| < 1$, and second, that $|u^*| = |k'Fx| < 1$ for all $t > 0$.

In conclusion then, the above relationships and remarks enable us to interpret the quadratic results to give the "singular" solutions to the optimal control problem as given in (1) and (2). We have that when x_0 satisfies $k'x_0 = 0$ and $|k'Fx_0| < 1$ then the minimum performance index (2) subject to the constraints (1) is given by $J^* = x_0'Px_0$ where P is given by (9) and (10). The optimal control u^* is $u^* = k'Fx$ with k' given as in (10). We note that the closed-loop system $\dot{x} = (I - gk')Fx$ is asymptotically stable. We also note that a consequence of the fact that $k'x = 0$ is that the optimal control law may be realized by a relay in its sliding mode, see Ref. [2]. That is, alternatively we have that $u^* = \text{sgn}(k'x)$.

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Résumé—Le problème de commande optimale duale consistant à trouver une commande u pour minimiser $J = \frac{1}{2} \int_0^\infty x'Qx dt$ avec les contraintes $Q > 0$, $\dot{x} = Fx + gu$, $|u| \leq 1$, est résolu pour ces solutions "singulières" en employant la théorie bien connue du régulateur quadratique. L'intérêt de la méthode, en dehors de celui de sa simplicité, est celui de pouvoir l'étendre facilement aux cas où Q est positif et semi-défini. Il est également tout à fait simple d'étendre les résultats au cas des entrées multiples et de la variation dans le temps.

Zusammenfassung—Das duale optimale Regelungsproblem der Suche einer Regelung u zur Minimierung von $J = \frac{1}{2} \int_0^\infty x'Qx dt$ mit den Beschränkungen $Q > 0$; $\dot{x} = Fx + gu$; $|u| \leq 1$ wird für seine "singulären" Lösungen unter Benutzung der wohlbekanntesten quadratischen Regelungstheorie gelöst. Die Bedeutung der Approximation liegt, abgesehen von ihrer Einfachheit darin, daß sie leicht auf weitere Fälle ausgedehnt werden kann, wenn Q positiv semidefinit ist. Es ist also naheliegend, die Resultate auf den Fall von zeitvariablen multiplen Eingängen zu erweitern.

Резюме—Проблема дуального оптимального управления заключающаяся в нахождении управления u для сведения к минимуму эшШоб9Их2т с огрничениями ИБЮ, 9ШЕхЖ34, .4.зЩ1, решается для ее "особенных" решений используя хорошо известную теорию квадратичного регулятора. Интерес этого метода, сверх его простоты, состоит в возможности легко обобщить его к случаям когда И положителен и частично определен. Также весьма просто обобщить результаты к случаю многочисленных входов и изменения по времени.