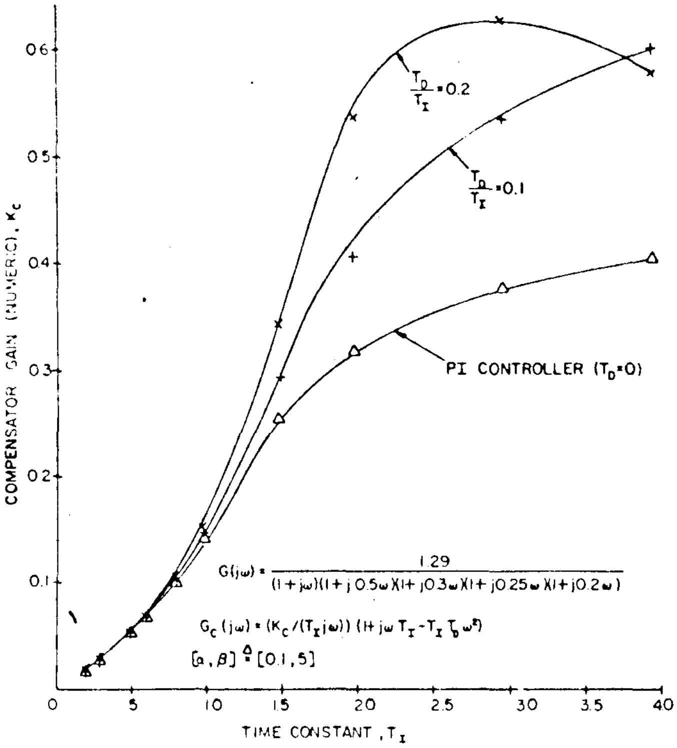

Fig. 3. Compensator parameters for system of Example 2 using circle criterion.

and the sector of nonlinearity be [0.1, 5]. It was proposed to introduce into the system a PID controller of the form given by (1). Also, it was desired to compare the designed PID compensator parameter pairs with those using a proportional plus integral (PI) control action.

The circle criterion was applied to the system of (10) with a PID controller included. For a PI action, the time constant ratio T_D/T_I was set equal to zero, thus eliminating the derivative action from the controller. Fig. 3 shows the design results.

Discussion and Conclusions

Compensator design results for a PID controller have been presented using both the analog simulation and a digital computer algorithm based on the circle criterion. Several T_D/T_I ratios were used for the study. It was noted that at lower T_I values the allowable compensator gain increased as T_D increased. However, at higher values of T_I, due to the crossover and parameter coupling effects in the PID controller, the reverse was true. Analog simulation for a specific nonlinearity located in a sector yielded an allowable gain much higher than that obtained by the application of the circle criterion.

The derivative action was found to have a marked influence on the overall system gain. The interaction effects in the PID controller appeared much later, as exhibited by a decrease in gain at higher T_I values. Also, beyond the crossover point, higher allowable gains were obtained at lower T_D/T_I ratios than at higher such ratios. Thus higher overall system gains would result if in a PID controller the T_D/T_I ratio were small (preferably less than 0.25) and the operating T_I value fell in a range below the crossover point.

Devendra P. Garg
Dep. Mech. Eng.
Massachusetts Institute of Technology
Cambridge, Mass. 02139

References

[1] D. G. Magiros, "Unification and classification of stability concepts of dynamic systems," presented at the 7th Int. Symp. Space Technology and Science, Tokyo, Japan, May 1967.
[2] L. M. Zoss, A. G. Witte, and J. E. Marsch, "Nonlinear design improves 3-mode controller," *Instrument. Tech.*, vol. 16, Feb. 1969, pp. 54–57.
[3] A. U. Meyer and J. D. Hsu, "The evaluation of frequency response stability criteria for nonlinear systems via logarithmic gain-phase plots," in *1968 Joint Automatic Control Conf., Preprints*, pp. 797–811.
[4] G. Zames, "On the input–output stability of time-varying nonlinear feedback systems—part II: conditions involving circles in the frequency plane and sector nonlinearities," *IEEE Trans. Automat. Contr.*, vol. AC-11, July 1966, pp. 465–476.
[5] R. W. Brockett, "The status of stability theory for deterministic systems," *IEEE Trans. Automat. Contr.*, vol. AC-11, July 1966, pp. 596–607.
[6] Y. Takahashi, M. Rabins, and D. Auslander, *Control and Dynamic Systems*. Reading, Mass.: Addison-Wesley, 1970, p. 94.
[7] D. Garg, D. Auslander, and M. Rabins, "A synthesis technique for nonlinear control systems using a frequency domain stability criterion," *Trans. ASME*, ser. D, vol. 93, June 1970, pp. 348–354.
[8] D. Garg and M. Rabins, "A parameter plane comparison of the Popov and circle criteria for nonlinear system synthesis," in *1970 Joint Automatic Control Conf., Preprints*, pp. 265–269.

Three-Term Controller Parameter Selection Using Suboptimal Regulator Theory

Abstract—**The classical problem of parameter selection of three-term controllers within the framework of suboptimal linear regulator theory is viewed. Iterative techniques are used to determine the controller parameters so that the expected value of a quadratic loss type performance index is minimized, with the initial states of the system a random variable uniformly distributed over a unit sphere.**

We consider single-input single-output linear time-invariant plants having state equations

$$\dot{x} = Fx + gu, \qquad x(t_0) \text{ given}$$
$$y = h'x. \tag{1}$$

The control u is assumed to be the output of a three-term controller with input y and is thus of the form

$$u = k_1 \int_0^t y\,dt + k_2 y + k_3 \dot{y} \tag{2}$$

where k_1, k_2, k_3 are the parameters of the three-term controller.

Certainly, there is a variety of classical control design methods available [1] which enable a selection of parameters $\{k_1, k_2, k_3\}$ to be made for any particular application. The purpose of this correspondence is to point out that this classical problem of selection $\{k_1, k_2, k_3\}$ may be solved using algorithms developed within the context of modern control. This is not a surprising result. However, the nontrivial nature of the problem when phrased in optimal control theory terms is surprising.

As a first step, we express the control law (2) as a state feedback law using (1). From (1) we have that

$$y = h'x, \qquad \dot{y} = h'Fx + h'gu, \qquad \ddot{y} = h'F^2x + h'Fgu + h'g\dot{u}$$

which means that (2), when differentiated, may be written as

$$(1 - k_3h'g)\dot{u} - (k_3h'F^2 + k_2h'F + k_1h')x - (k_3h'Fg + k_2h'g)u = 0.$$

Using the notation $\hat{k}$ as a normalization of k as follows:

$$\hat{k}' = [\hat{k}_1\hat{k}_2\hat{k}_3]' = (1 - k_3h'g)^{-1}[k_1k_2k_3]' \tag{3}$$

or

$$\hat{k} = ck'$$

where c is a scalar. This control law may be written as

$$\dot{u} = \hat{k}'[h \quad F'h \quad (F')^2h]'x + \hat{k}'[0 \quad g'h \quad g'F'h]'u \tag{4}$$

Equation (4) is an "output" feedback law in the sense that constrained state feedback is used.

Manuscript received June 3, 1970; revised August 10, 1970. This work was supported by the Australian Research Grants Committee.

We now observe that this law is nothing other than a state feedback law for the system (1) augmented with an integrator at its input [2]. That is, (4) may be written as

$$u_a = k_a' x_a$$

where

$$u_a = \dot{u}, \qquad x_a = \begin{bmatrix} x \\ u \end{bmatrix}$$

$$k_a = \begin{bmatrix} h & F'h & (F')^2 h \\ 0 & g'h & g'F'h \end{bmatrix} \hat{k} = T\hat{k}. \tag{5}$$

The augmented system equations are given from (1) and (5) as

$$\dot{x}_a = F_a x_a + g_a u_a \tag{6}$$

where

$$F_a = \begin{bmatrix} F & g \\ 0 & 0 \end{bmatrix}, \qquad g_a = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{7}$$

So far, all we have done is to show how the three-term controller can be viewed as a state-variable feedback law for the original system augmented with an integrator at its input. If the gain elements k_1, k_2, k_3 of a three-term controller are specified, then $\hat{k}$ may be calculated using (3), and the state feedback gain k_a for the augmented system (7) can be calculated using (5).

The three-term controller problem becomes an optimal problem once a performance index for the augmented system (6) is defined. A convenient index to use [3] is simply

$$\bar{V} = \mathrm{tr}\,[\bar{P}]$$

where

$$\bar{P} = \int_0^\infty \exp\,[(F_a + g_a k_a')' t](Q + k_a k_a') \exp\,[(F_a + g_a k_a')t]\,dt$$

for some positive-definite symmetric Q. The three parameters k_1, k_2, k_3 may be adjusted in some systematic way until this index is minimized. (An interpretation of the index is that $\bar{V} = E \int_0^\infty (x_a' Q x_a + u_a^2)\,dt$, where $x(t_0)$ is a *random* variable *uniformly* distributed on the unit sphere, i.e., $E[x(t_0)x'(t_0)] = (1/n)I$, where E denotes expected value and n is the dimension of the state vector.) It can be shown that

$$\frac{d\bar{V}}{dk} = cT'M[cTk + Pg_a]$$

where P satisfies

$$P(F_a + cg_a k'T') + (F_a + cg_a k'T')'P + Q + c^2 Tkk'T' = 0$$

and M satisfies

$$M(F_a + cg_a k'T') + (F_a + cg_a k'T')'M + I = 0.$$

This information can be used in a gradient procedure to converge to the optimum K^*.

For the case when

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -9 & -10 \end{bmatrix}, \qquad g = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$h' = [1 \quad 0 \quad 0], \qquad Q = I$$

the optimal feedback law has been calculated to be

$$k_1 = 0.21, \qquad k_2 = 2.21, \qquad k_3 = 1.38.$$

The minimum index is $\bar{V}^* = 7.64$. (This may be compared to the value $\bar{V}^* = 6.31$, which is achieved by the augmented system with no constraints on k_a—the standard regulator case. See [4] for the case when the three-term controller is the standard optimum controller for the augmented system.) The system response is shown in Fig. 1.

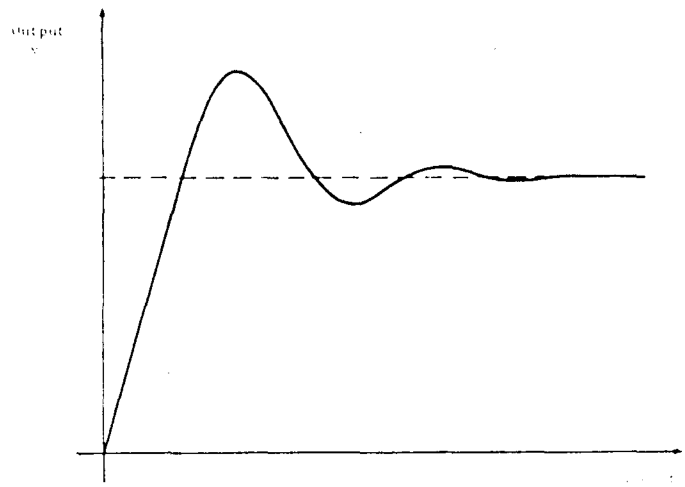

Fig. 1. System response for suboptimally designed three-term controller

It is somewhat surprising that so much theory is necessary in order to "optimally" select the parameters of a three-term controller, particularly since in practice three-term controller parameters are often selected for quite complex systems with very little effort and quite good results [1].

DARRELL WILLIAMSON
JOHN B. MOORE
Dep. Elec. Eng.
University of Newcastle
New South Wales, 2308, Australia

REFERENCES

[1] L. B. Koppel, *Introduction to Control Theory, with Application to Process Control.* Englewood Cliffs, N.J.: Prentice-Hall, 1968.
[2] J. B. Moore and B. D. O. Anderson, "Optimal linear control systems with input derivative constraints," *Proc. Inst. Elec. Eng.*, vol. 114, Dec. 1967, pp. 1987–1990.
[3] W. S. Levine and M. Athans, "On the determination of the optimal constant output feedback gains for linear multivariable systems," *IEEE Trans. Automat. Contr.*, vol. AC-15, Feb. 1970, pp. 44–48.
[4] N. N. Puri, "Optimal design of regulators," in *Proc. 3rd Annu. Allerton Conf. Circuit and System Theory*, 1965.

Redundancy in Linear Optimum Regulator Problem

Abstract—**The optimum linear regulator problem is the determination of the input to $\dot{x} = Fx + Gu$ that minimizes $\int_0^\infty x^T Q x + u^T u\,dt$, and the well-known solution is a feedback law $u = -K^T x$. It is known that the problem statement is redundant, in that distinct matrices Q_1 and Q_2 can yield the same feedback law K. Such matrices are called equivalent, and a simple test for equivalence is available for the single-input case. This note generalizes the test to the multivariable case.**

INTRODUCTION

Consider the linear time-invariant system

$$\dot{x} = Fx + Gu \tag{1a}$$

where F is $n \times n$, G is $n \times m$, and (F, G) is controllable. The normalized optimum regulator problem is the determination of the control function $u(t)$ that minimizes the index

$$J = \int_0^\infty x^T Q x + u^T u\,dt. \tag{2}$$

Manuscript received July 13, 1970; revised November 2, 1970.