Formations of Autonomous Agents: Rigidity Maintenance and Formation Operations

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Outline

- Aim of Seminar
- Formations and Problem Overview
- Rigid Formations
- Directed Formations and Persistence
- Minimal Cover Problem
- Splitting and Merging Formations
- Closing Ranks
- Recent Developments and Open Problems

Aim of Seminar

- Introduce the concept of a formation, especially a rigid formation and persistent formation
- Review results on rigidity and persistence
- Describe problems of splitting formations, merging formations and ‘closing ranks’

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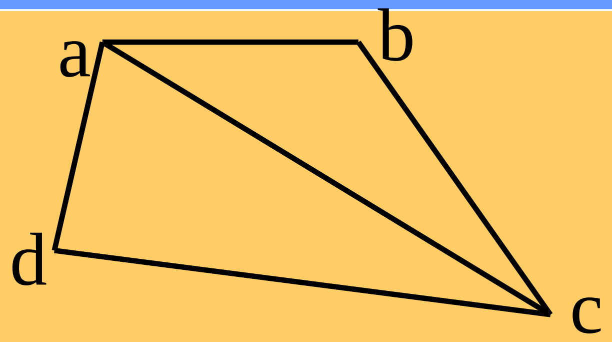
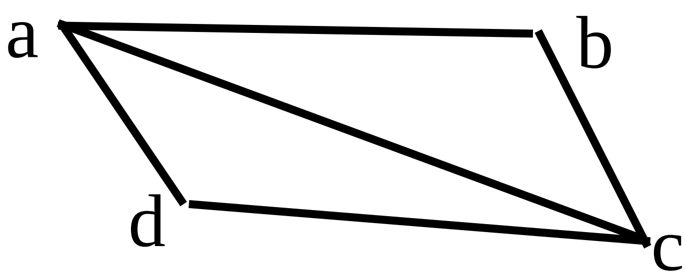
Formations

- A formation is a collection of agents (point agents for us) in two or three dimensional space
- A formation is ***rigid*** if the distance between each pair of agents does not change over time
- In a rigid formation, normally only ***some distances*** are explicitly maintained, with the rest being consequentially maintained.

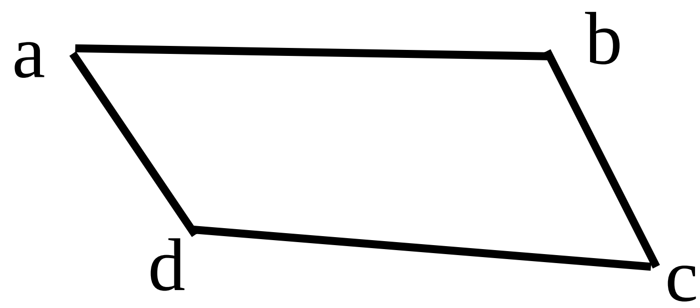


The distances ab,bc,cd,ad and ac are explicitly maintained and the distance bd is maintained as a result of the topology.

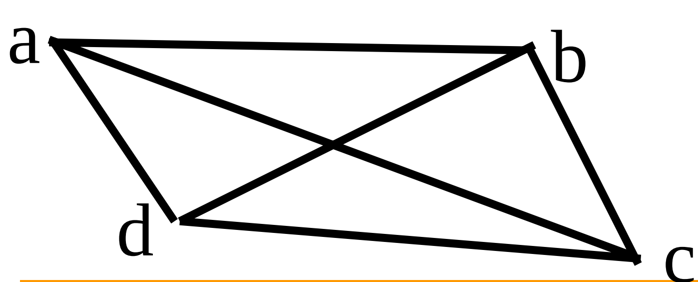
Rigid and Nonrigid Formations



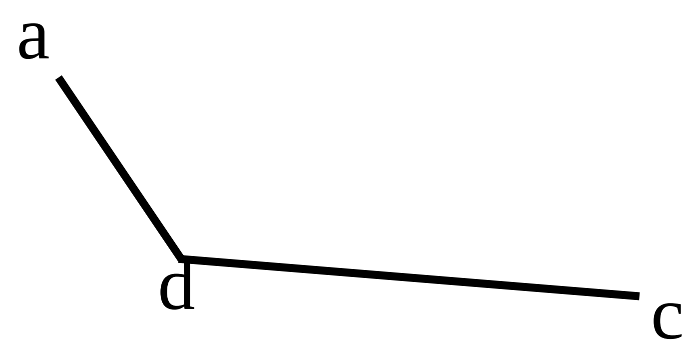
MINIMALLY RIGID



NONRIGID



RIGID, BUT NOT MINIMALLY SO

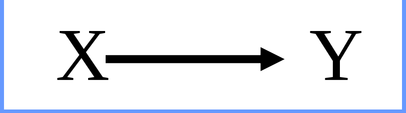


NONRIGID

Directed Graphs

- Maintaining a formation shape is done by maintaining certain inter-agent distances
- If the distance between agents X and Y is maintained, this may be:
  - A task jointly shared by X and Y, or
  - Something that X does and Y is unconscious about, or conversely (leader/follower concept)
- **Undirected** graphs model the first situation. **Rigid graph theory** is applicable.

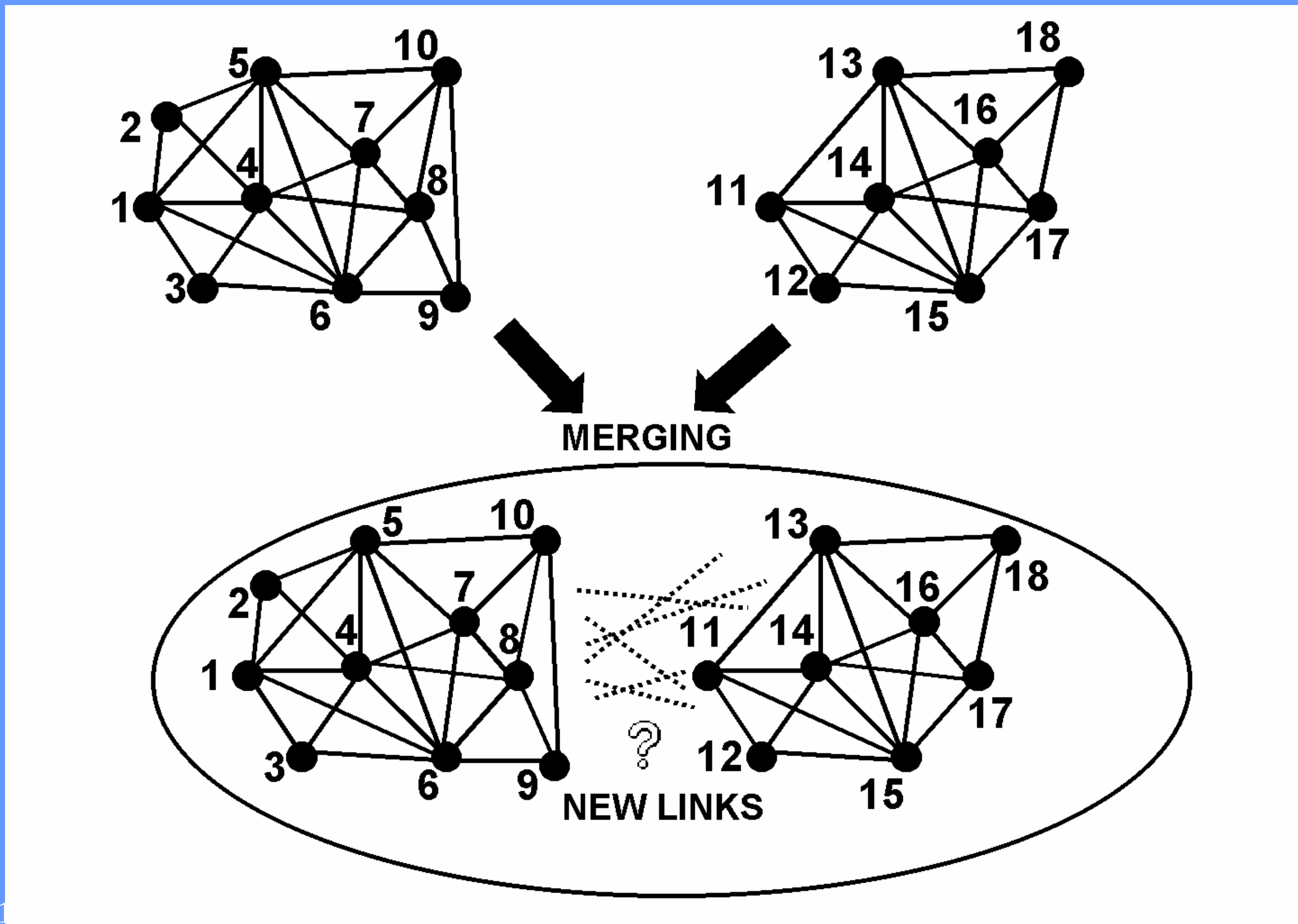
- **Directed** graphs model the second situation. All the rigidity type questions and theories have to be validated **and/or modified** with new results for directed graphs.



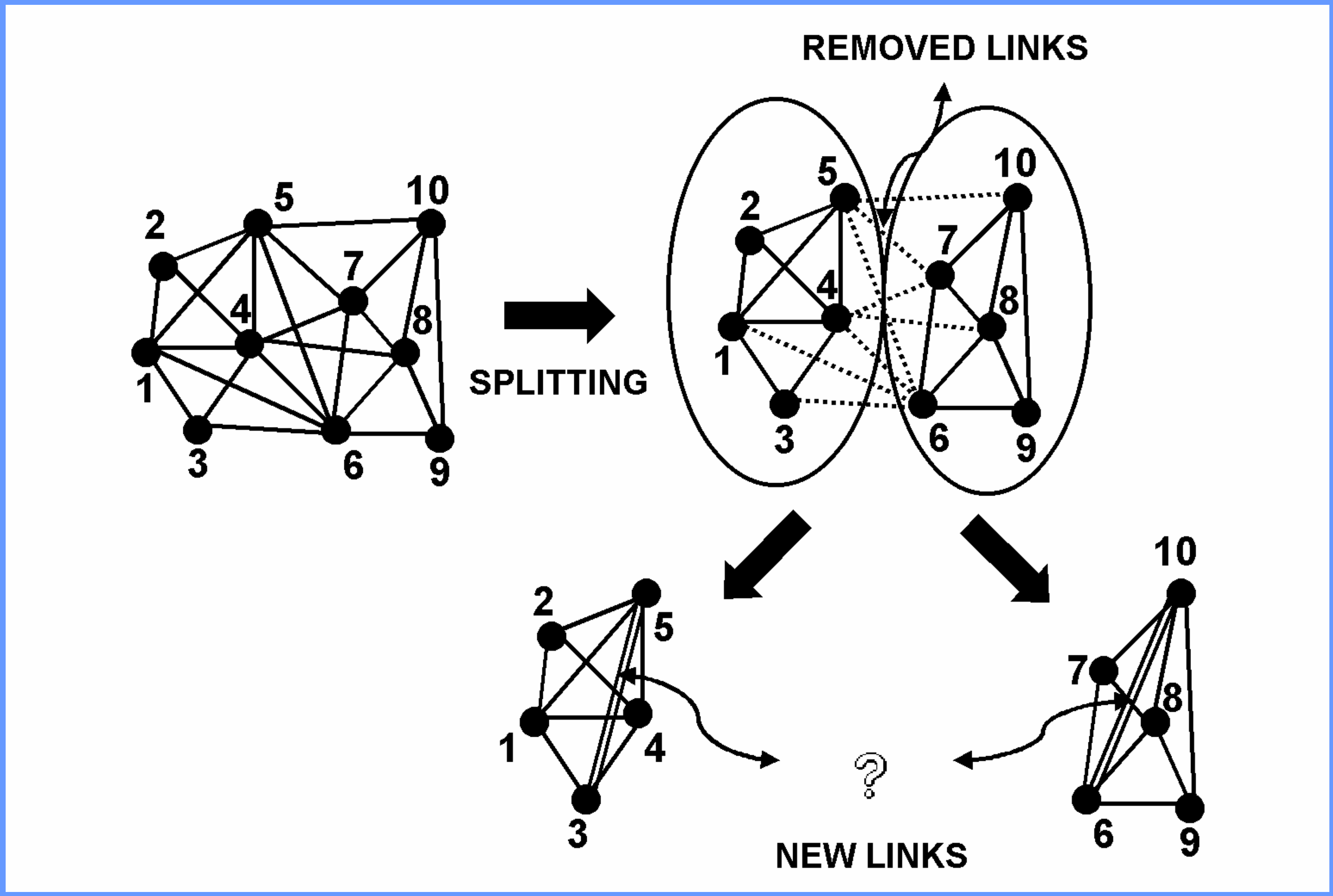
Formations and Operations

- The topologies of formations may have to be changed dynamically with (diagrams next slides)
  - ❑ Merging
  - ❑ Splitting
  - ❑ Closing ranks
- We aim for rigidity after the operation--which may require some new links to be explicitly established
- It is natural to preserve explicit distance links to the extent possible from before the change to after the change
- No distinction at this point between directed/undirected

Formation Merging

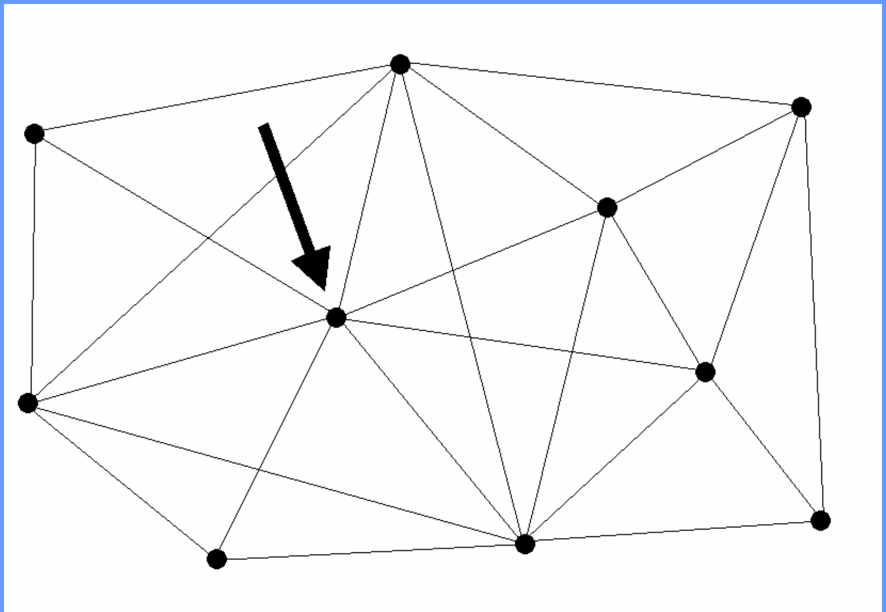


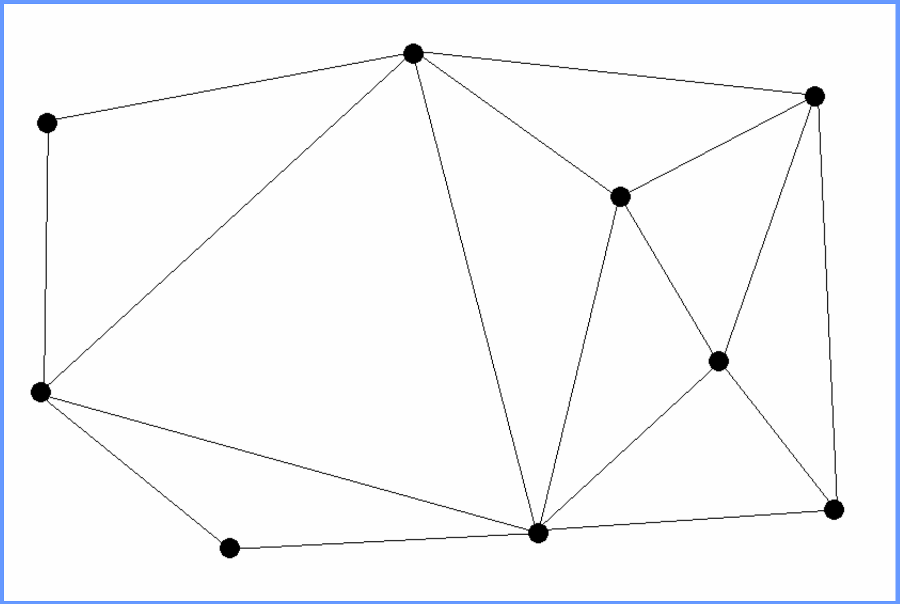
Formation Splitting



Closing Ranks

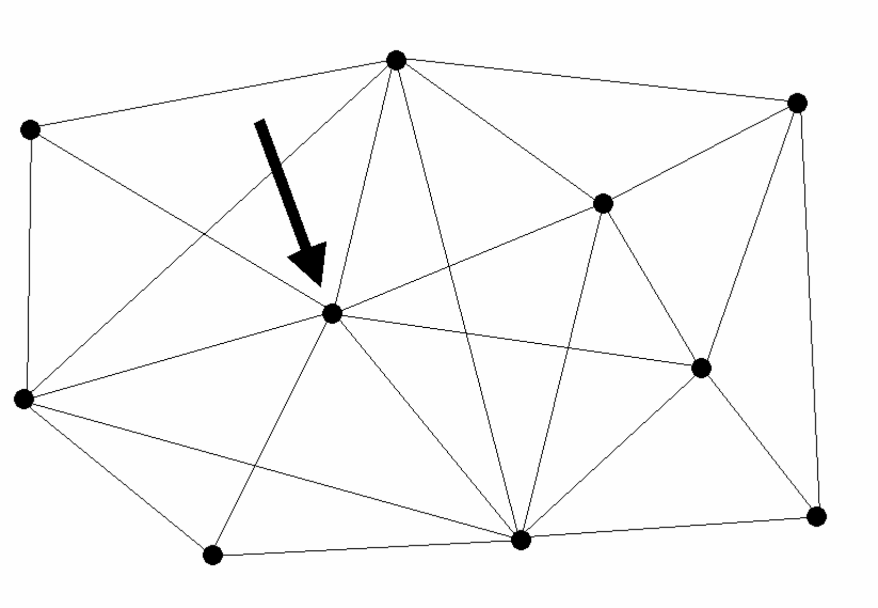
- One (***or more***) agents is removed, generally destroying rigidity
- Diagram depicts ***three-dimensional*** formation losing one agent and its links

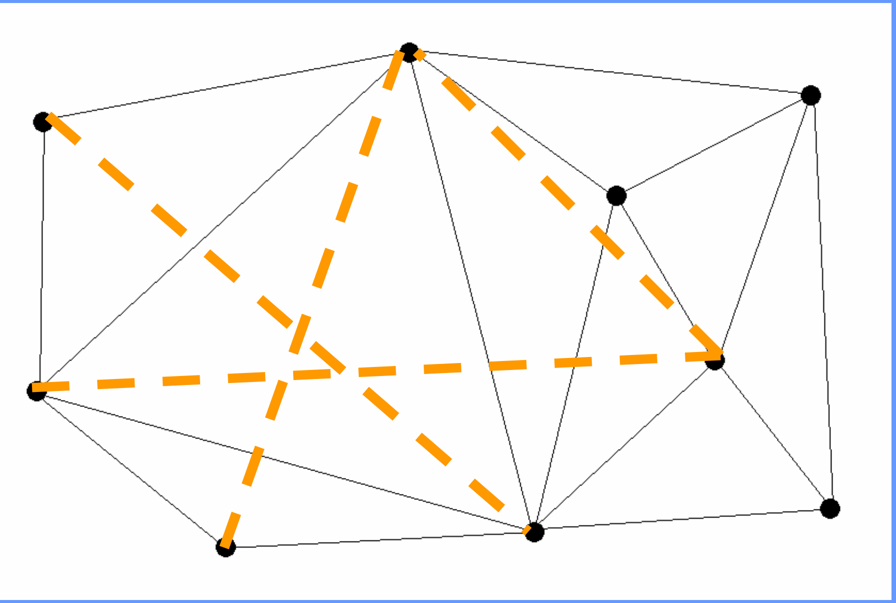




Closing Ranks

- One (***or more***) agents is removed, generally destroying rigidity
- Diagram depicts ***three-dimensional*** formation losing one agent and its 7 links
- Remaining links kept and 4 new ones added giving rigidity





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Summary of next 6 slides

- Rigidity (with undirected graph) of a two dimensional structure can be characterised with
  - Linear algebra
  - Graph theory
- Rigidity of a three dimensional structure can be
  - Characterised with linear algebra
  - Partially characterised with graph theory
- A ***graph*** is defined by a set of vertices V and a set of edges E joining certain vertex pairs

Rigid Formations

- Consider a point formation $F = (\{p_1, p_2, \ldots p_n\}, L)$ with m maintenance links defined by $(i,j) \in L$, moving along a trajectory $p(t)$ with each distance $d_{ij} = \| p_i - p_j \|$ constant. Along such a trajectory:

$$(p_i - p_j)^T \frac{d}{dt}(p_i - p_j) = 0 \ \ \forall \ (i,j) \in L$$

Rigid Formations

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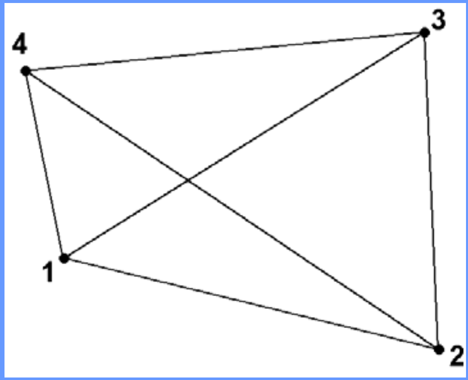
$$(p_i - p_j)^T \frac{d}{dt}(p_i - p_j) = 0 \;\; \forall\, (i,j) \in L$$

- These scalar equations may be gathered as:

$$R(p)\frac{dp}{dt} = 0$$

where $R(p)$ is $m \times dn$ (d = underlying space dimension and n = number of vertices.)

Rigid Formations



Sample two dimensional Rigidity Matrix--a ***Matrix Net*** $\sum x_i M_i + y_i N_i$ in coordinates of points.

	v_1	v_2	v_3	v_4
(1,2)	$x_1 - x_2 \quad y_1 - y_2$	$x_2 - x_1 \quad y_2 - y_1$	0	0
(1,3)	$x_1 - x_3 \quad y_1 - y_3$	0	$x_3 - x_1 \quad y_3 - y_1$	0
(1,4)	$x_1 - x_4 \quad y_1 - y_4$	0	0	$x_4 - x_1 \quad y_4 - y_1$
(2,3)	0	$x_2 - x_3 \quad y_2 - y_3$	$x_3 - x_2 \quad y_3 - y_2$	0
(2,4)	0	$x_2 - x_4 \quad y_2 - y_4$	0	$x_4 - x_2 \quad y_4 - y_2$
(3,4)	0	0	$x_3 - x_4 \quad y_3 - y_4$	$x_4 - x_3 \quad y_4 - y_3$

Rigid Formations

- In a **rigid** formation, the only motions possible are translation (2 or 3 directions) and rotation (one or three axes). Hence nullspace of $R(p)$ has dimension 3 or 6.
- $R(p)$ has $2n$ or $3n$ columns
- Theorem: Assume F is a formation with $n \geq d + 1$ points in d-space. F is rigid in d-space iff

$$\text{rank } R(p) = \begin{cases} 2n - 3 & \text{if } d = 2 \\ 3n - 6 & \text{if } d = 3 \end{cases}$$

- Note that R has the same rank for all p except a set of measure zero. So almost all formations with the same graph are either rigid or not rigid. We can speak of the graph being *generically rigid* or not.

Rigid Formations

- Laman's theorem offers ***a combinatorial characterization of rigidity***: in order for a graph $G = (V,L)$ to be generically rigid in 2-space, there must be a subset L' of L satisfying the following two conditions:
  1. $|L'| = 2|V| - 3$
  2. For any $V'' \subset V$ with $|V''| \geq 2$, and with L'' denoting the edges in L' joining vertices of V'', there holds $|L''| \leq 2|V''| - 3$

Rigid Formations

- Laman's theorem offers ***a combin*** [illegible] ***characterization of rigidity***: in [illegible] graph $G = (V,L)$ to be genericall[illegible] space, there must be a subset [illegible]ying the following two con[illegible]
  1. $|L'| = 2|V|$ [illegible]
  2. For a[illegible]h $|V''| \geq 2$, and with L'' de[illegible]ges in L' joining vertices of V'', t[illegible]olds $|L''| \leq 2|V''| - 3$

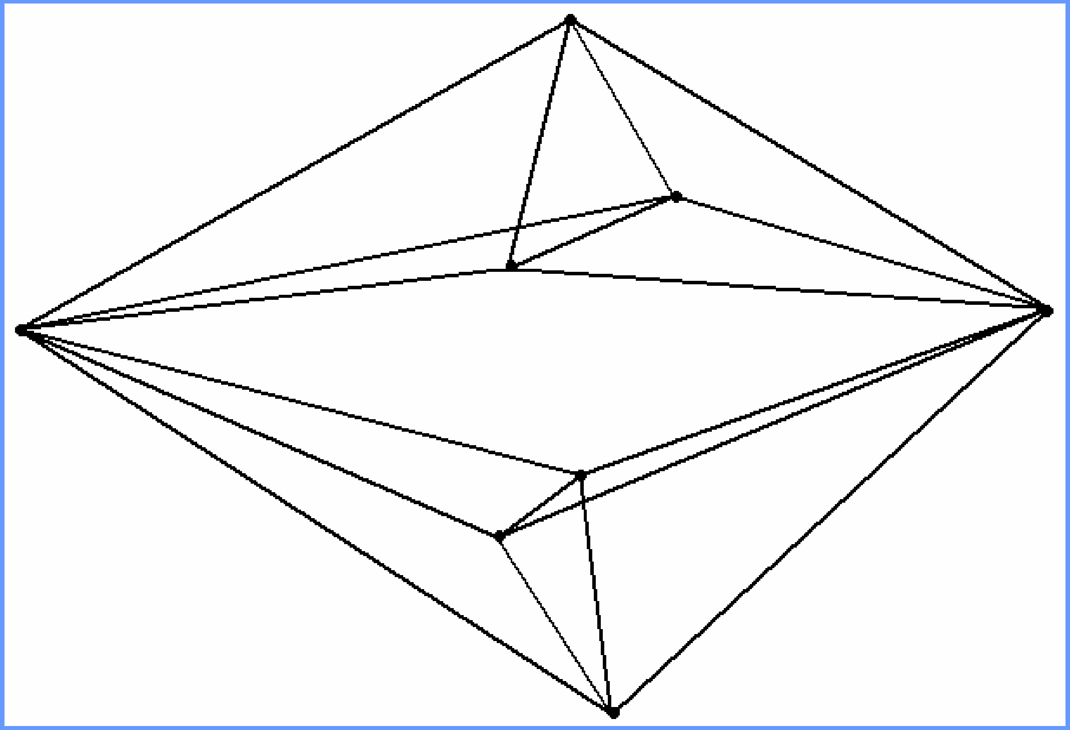
ALL ABOUT COUNTING
OF VERTICES AND EDGES

Rigid Formations

- Laman's theorem offers ***a combinatorial characterization of rigidity***: in order for a graph $G = (V,L)$ to be generically rigid in 2-space, there must be a subset L' of L satisfying the following two conditions:
  1. $|L'| = 2|V| - 3$
  2. For any $V'' \subset V$ with $|V''| \geq 2$, and with L'' denoting the edges in L' joining vertices of V'', there holds $|L''| \leq 2|V''| - 3$
- Laman conditions can be checked in ***polynomial time***
- ***There is no comparable result for 3-space***. The generalizations of the Laman counts are necessary but not sufficient.
- There is in fact ***an independent necessary condition*** in 3-space: G is 3-connected.

3-space Laman counterexample

- There is no comparable result for 3-space. The generalizations of the Laman counts are necessary but not sufficient.



Minimal Rigidity

- Recall that $R(p)$ is $m \times dn$ (m = number of edges, d = underlying space dimension and n = number of vertices.)
- Recall the rigidity condition:

$$\text{rank } R(p) = 2n - 3 \text{ if } d = 2$$
$$3n - 6 \text{ if } d = 3$$

- ***Minimally rigid*** structures $\Leftrightarrow$ take away any edge and you lose rigidity $\Leftrightarrow$ the row count of a maximal rank $R(p)$ is exactly $2n - 3$ if $d = 2$ or $3n - 6$ if $d = 3$ $\Rightarrow$ vertex count is $2n - 3$ if $d = 2$ or $3n - 6$ if $d = 3$
- If an $R(p)$ has rank 2n-3 or 3n - 6, you cannot increase the rank by adding more rows $\Leftrightarrow$ once a structure is rigid, you can add more edges and it obviously stays rigid

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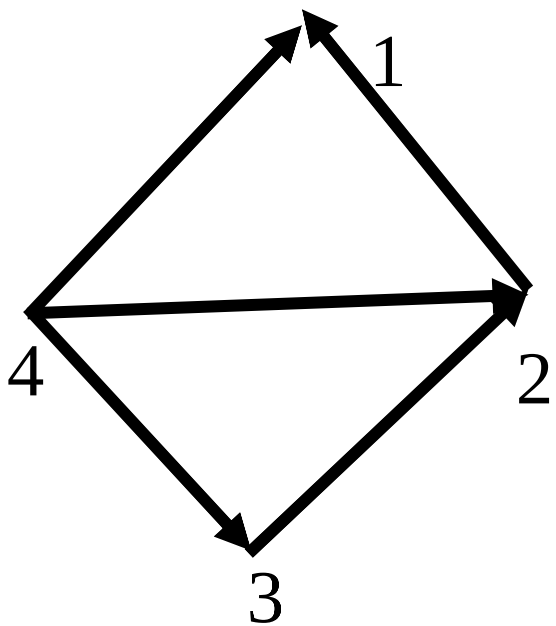
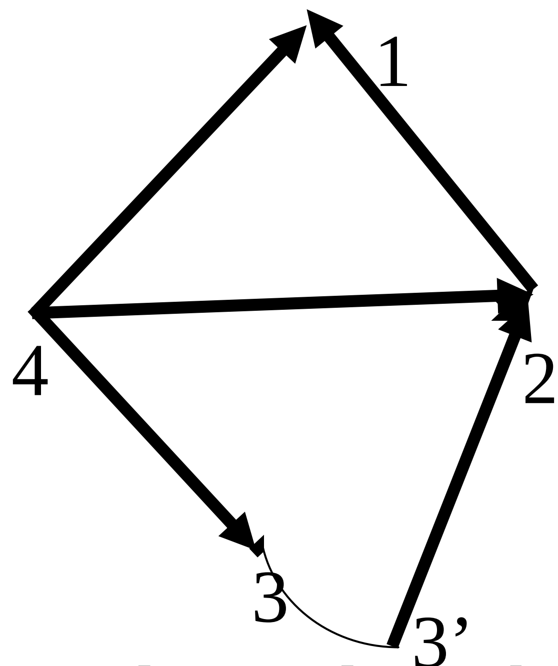
Rigidity is not enough!



Undirected graph is rigid!

- Agent 1 is unconstrained (leader), and agent 2 must follow agent 1, and agent 3 must follow agent 2.
- Agent 3 can move on a circle, even if agent 1 and agent 2 are stationary.

Rigidity is not enough!



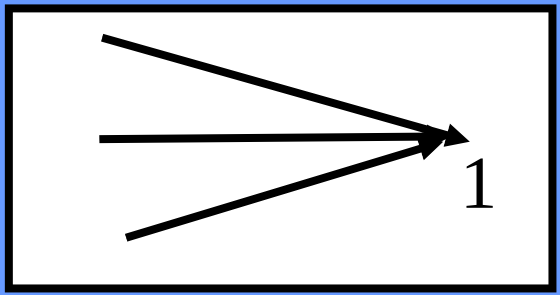
- Agent 1 is unconstrained (leader), and agent 2 must follow agent 1, and agent 3 must follow agent 2.
- Agent 3 can move on a circle, even if agent 1 and agent 2 are stationary.
- Agent 4 can no longer maintain all three distances

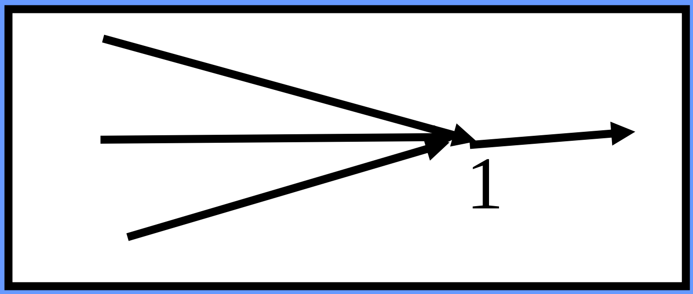
Set-up is not ***constraint consistent.***

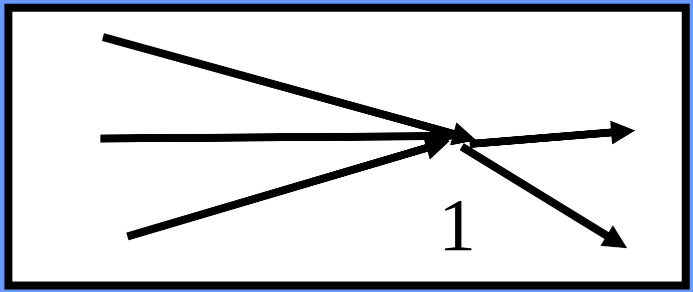
Directed graph generalization

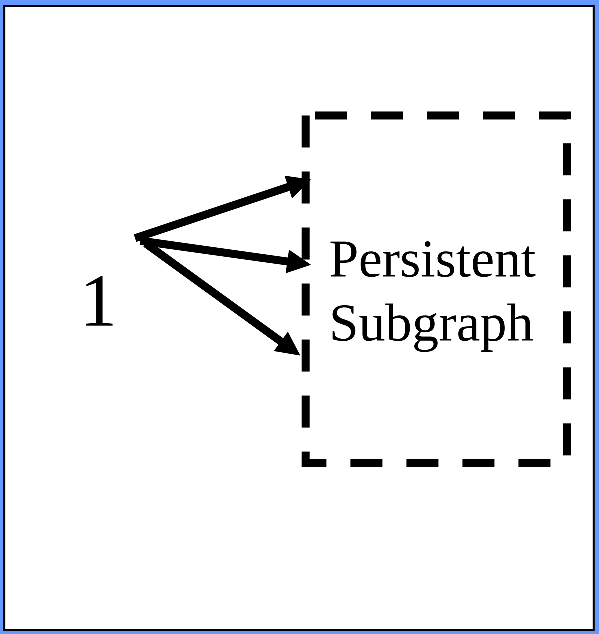
- Rigidity says shape maintained ***if*** distances are maintained; constraint consistence for a vertex says distances ***can*** be maintained
- Formation maintenance requires a directed graph to be both rigid and constraint consistent. We call this ***persistence***
- In 2D, out-degree vertices of degree 0, 1 or 2 are always constraint consistent. Degree 3 vertices may or may not be constraint consistent.

Constraint Consistence









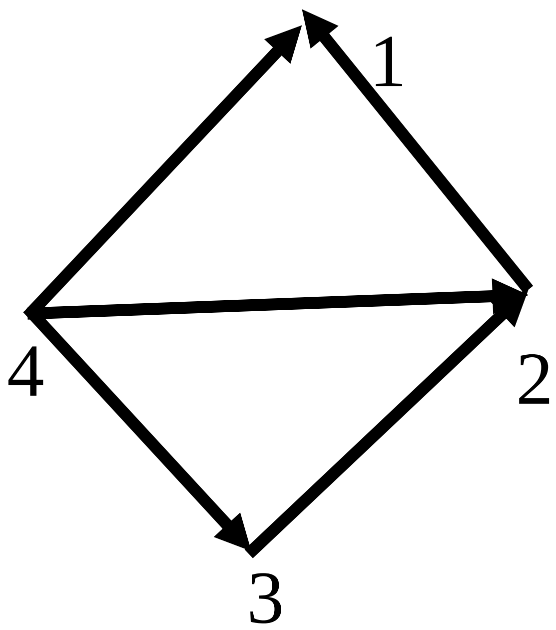
Node 1 is always constraint consistent

2D formation--directed graph

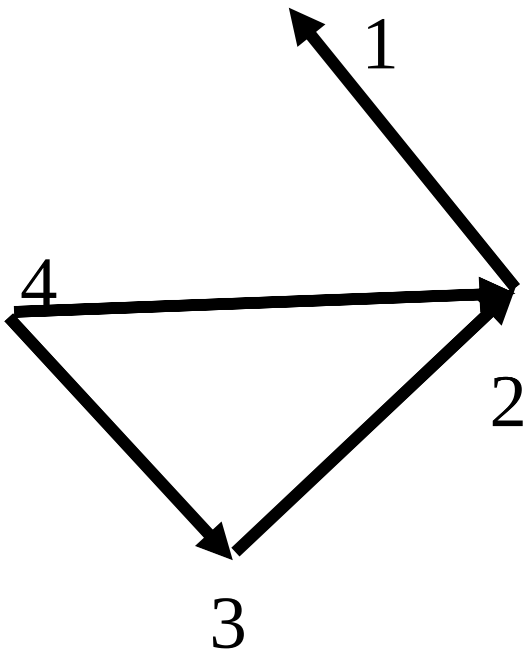
Theorem: A 2D directed graph is persistent if and only if every graph is rigid which is obtained by deleting edges from a vertex of out-degree 3 or more until only two outgoing edges are left.

- Converts the problem of checking persistence to one of checking rigidity
- Having a persistent graph in two dimensions is necessary and sufficient to maintain formation shape

Rigidity is not enough!



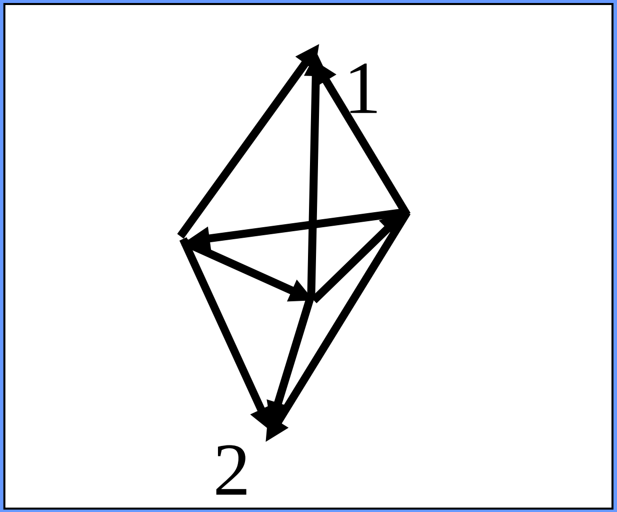
Undirected graph is rigid!



Nonrigid graph: 4 has out-degree 3, 14 is removed

3D formation-directed graph

- Rigidity and constraint consistence (i.e. persistence) are both necessary.
- They are not sufficient; each agent may be constraint consistent, but collections may not be!



- Graph is rigid
- Graph is persistent
- Two unconstrained leaders are present--agents 1 and 2

3D formation-directed graph

- **Structural persistence** means: rigidity **plus** constraint consistence of individual vertices **plus** constraint consistence of **subsets** of vertices
- **Formation shape in any dimension is maintained with structural persistence**
- In 2D, structural persistence = persistence
- In 3D, structural persistence = persistence + do not have two unconstrained leaders.
- In 3D, persistence plus cycle-free (leader-follower style) implies structural persistence

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Minimal Cover Problem

- The ***minimal cover*** problem is to find a set of new edges to be inserted into the post operation (merging, etc) graph (or each subgraph in splitting problem) so that it is minimally rigid.
- There is a long-standing theory for undirected graphs for growing minimally rigid graphs, called Henneberg sequence theory.
  - In two dimensions, all minimally rigid graphs can be 'grown' by adding one vertex at a time to an existing minimally rigid graph following one of two standard procedures, and starting with a triangle of three vertices.
  - The theory is not as complete for three dimensions
  - A directed graph version has recently been developed.
- The minimal cover problem is solved by a variant of Henneberg sequence theory.

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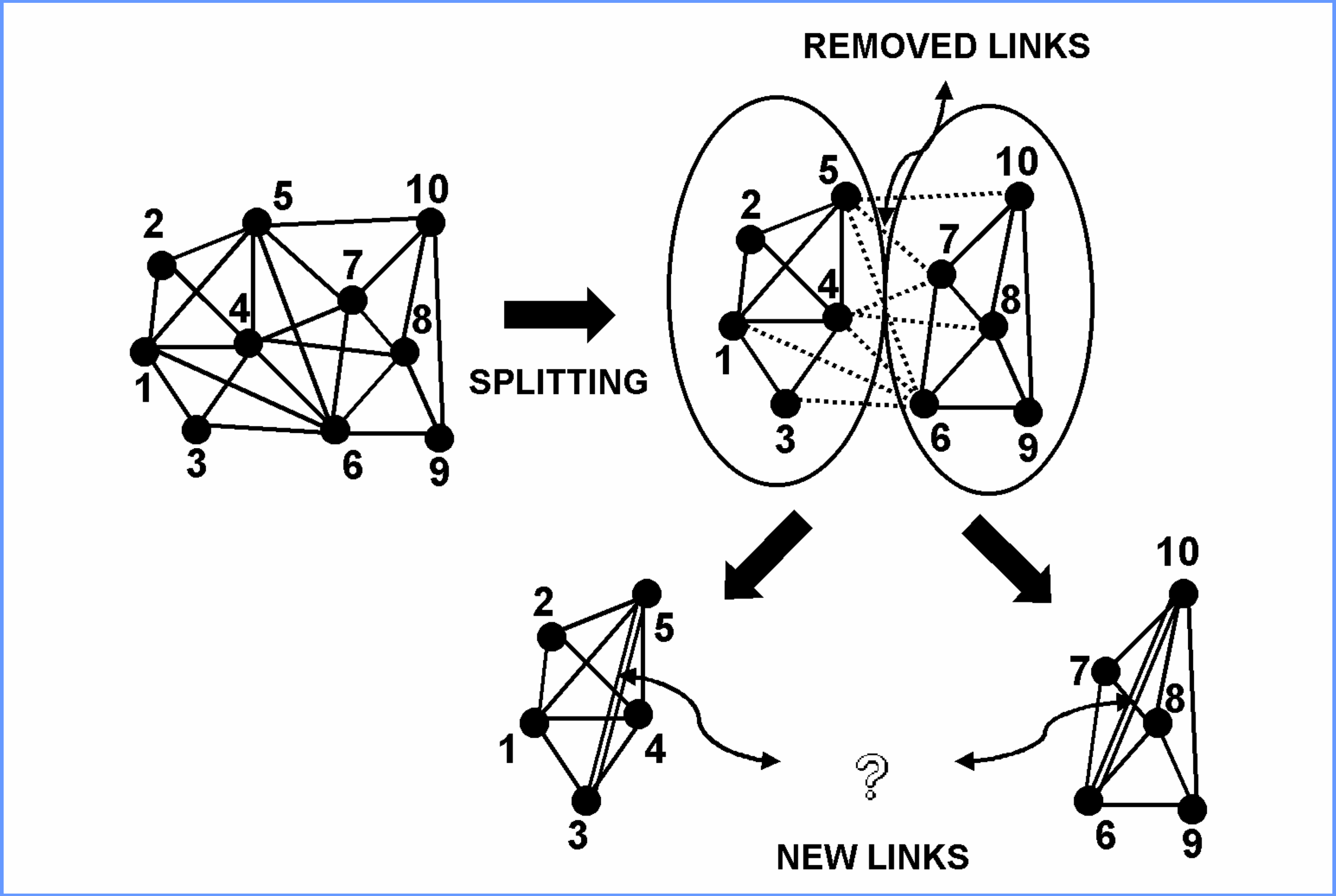
Formation Splitting



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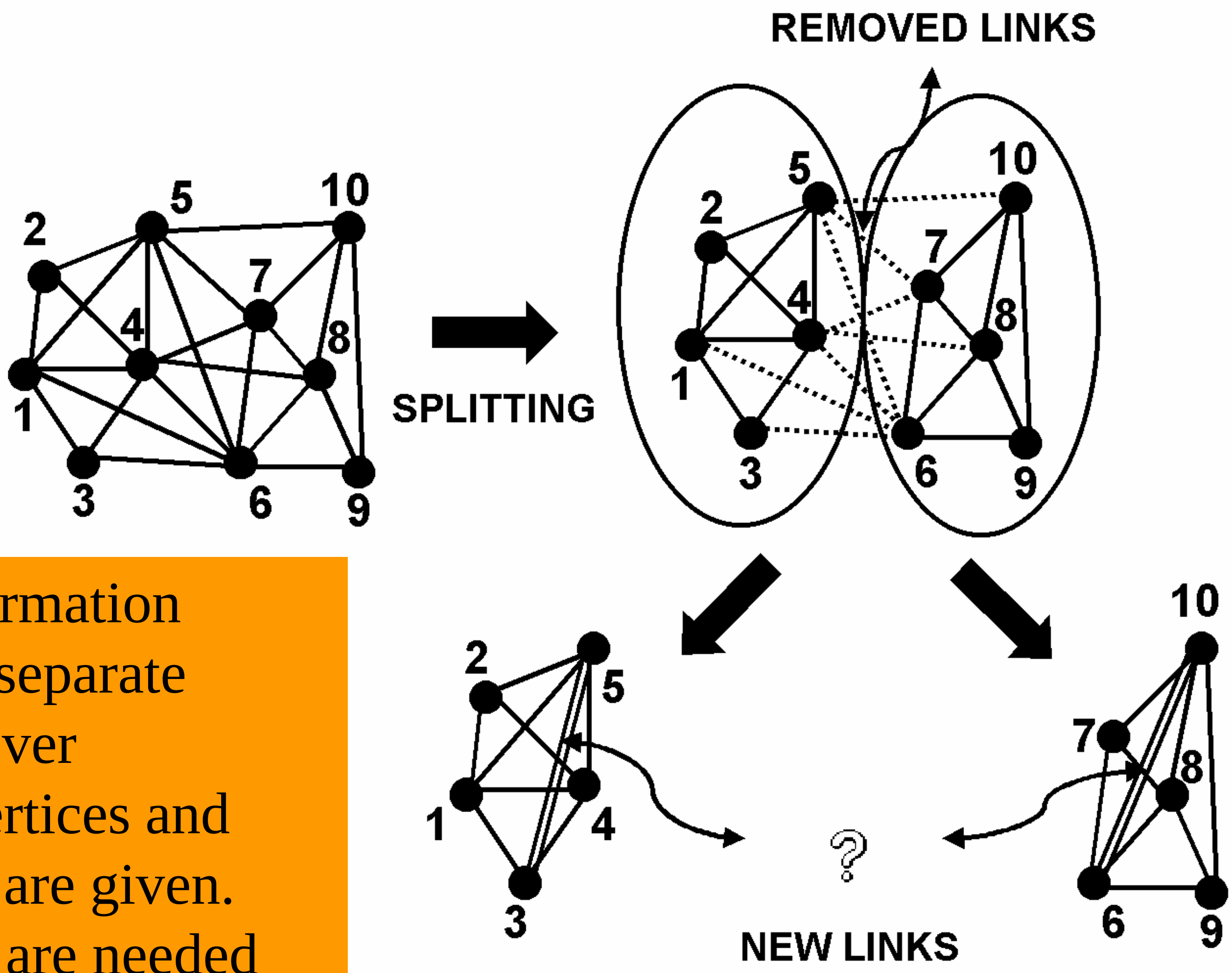
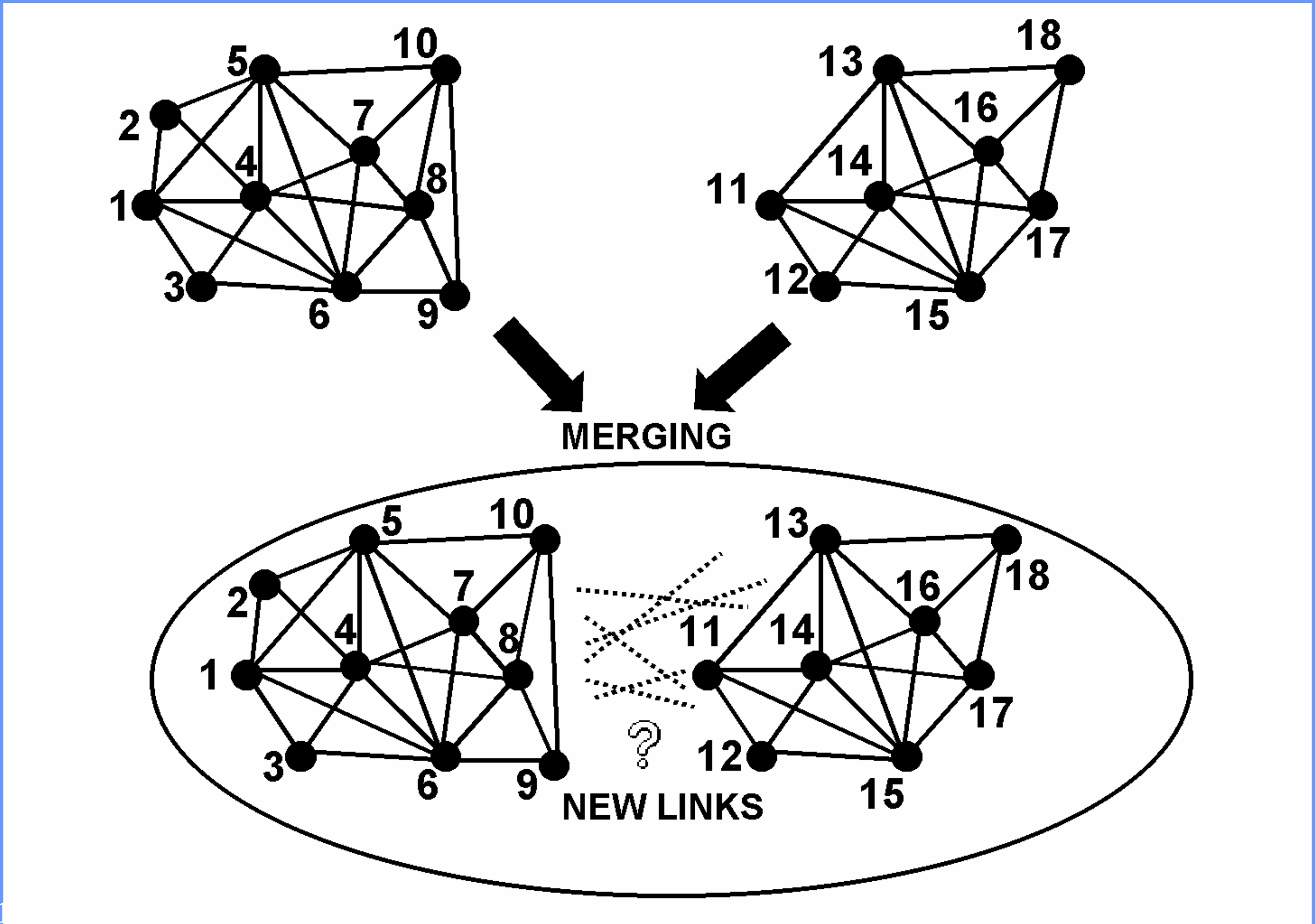


Each subformation provides a separate minimal cover problem;vertices and some links are given. New links are needed

Formation Merging



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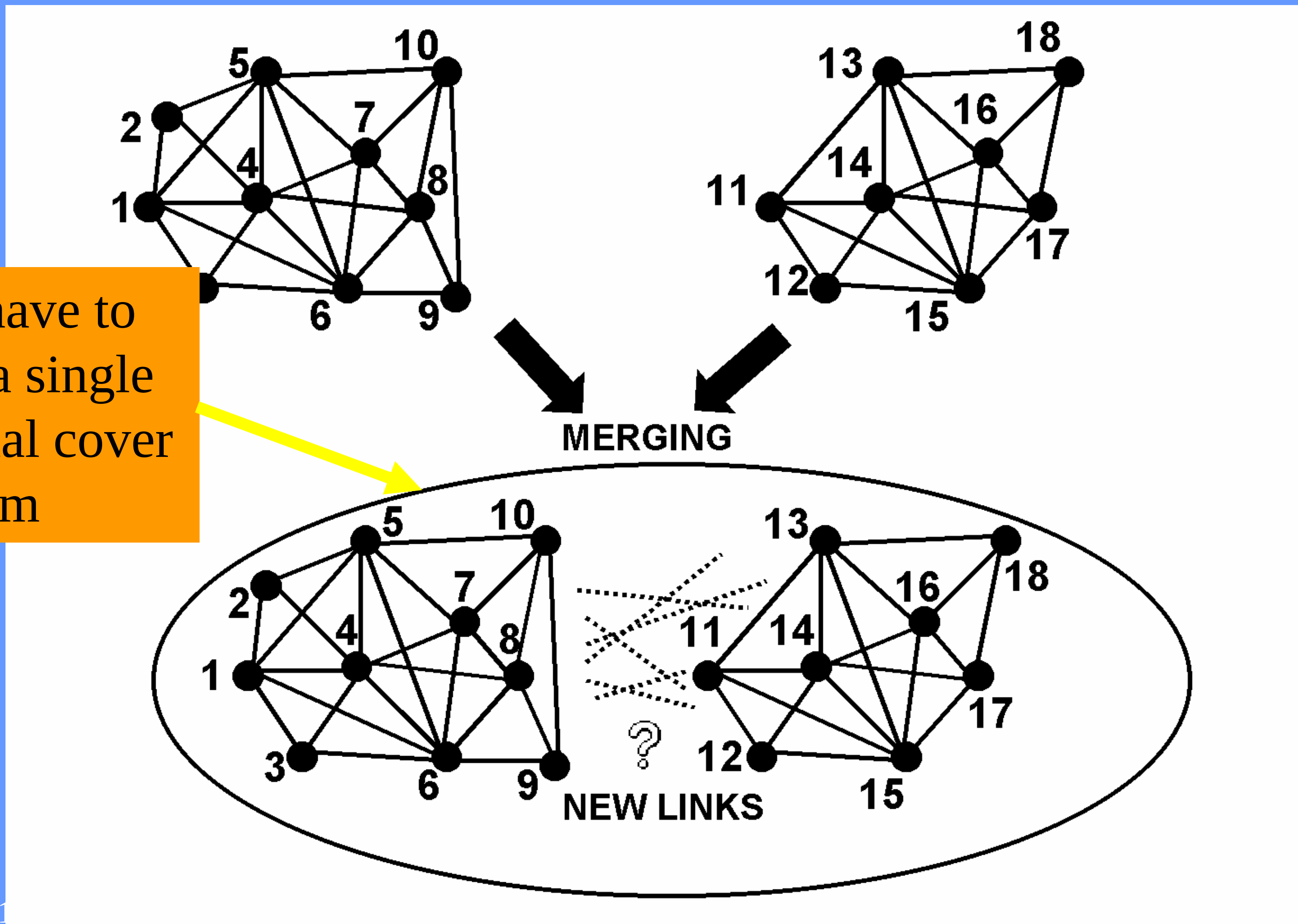

Now have to solve a single minimal cover problem

Formation Merging: Another view

- Think of two separate formations as **meta vertices** that are internally rigid--but forget the internal structure. A meta vertex has a position, its centre of gravity, and **an orientation.**
- Formation merging is about connecting vertices that are point vertices **with orientation**
- Using this view, it comes easily that two formations in two dimensions can be merged using three connecting links, with suitably chosen directions in the directed graph case.
  - Rules apply to selection of end vertices of new links.
- In three dimensions, six links are needed.
- One deal easily with merging three or more formations.

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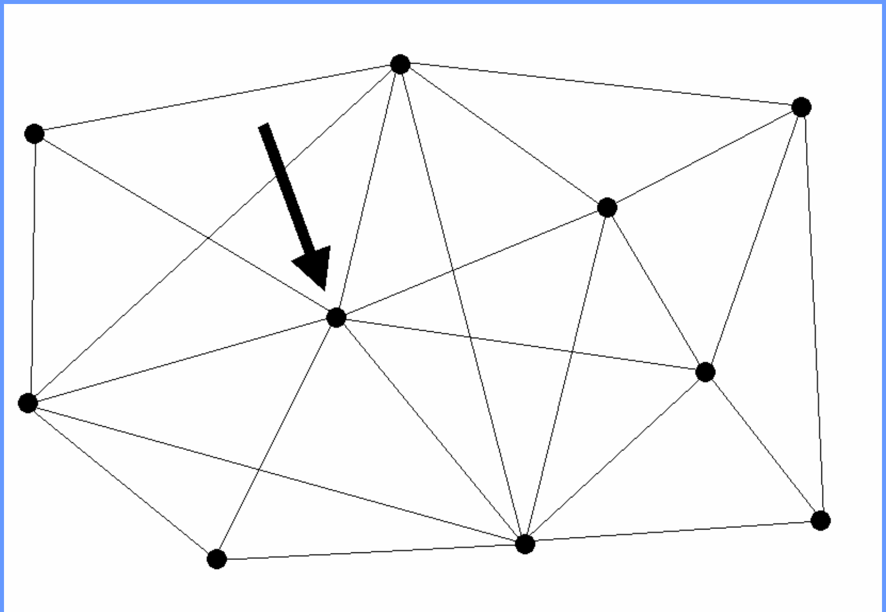
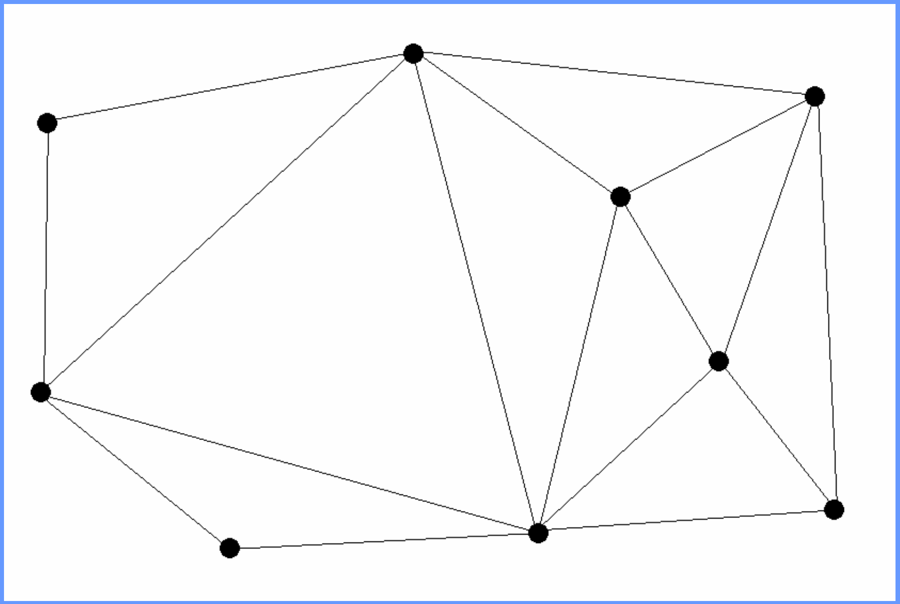
Closing Ranks

- One (***or more***) agents is removed, destroying rigidity
- Diagram depicts ***three-dimensional*** formation losing one agent and its 7 links





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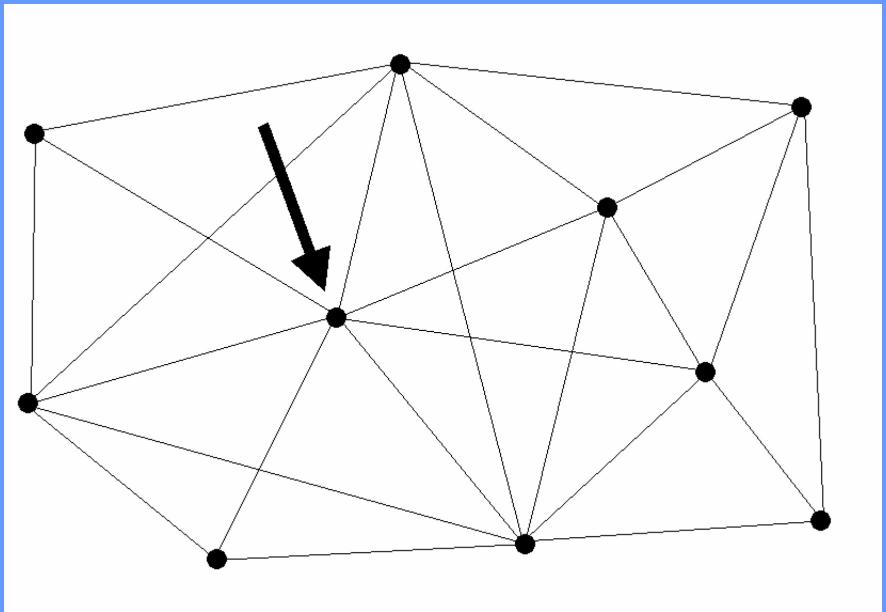

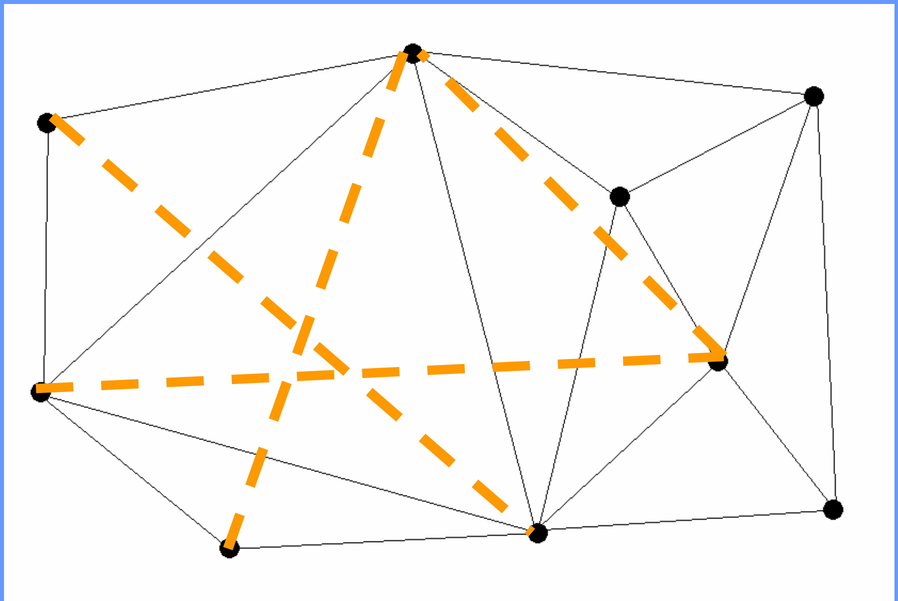

Closing Ranks

- In two (three) dimensions, *k* degree vertex removal requires *k-2 (k-3)* new edges.
- ***Key Conclusion 1:*** Closing ranks can always be achieved when one vertex with its incident edges is lost by making **connections among neighbours of the lost vertex**
- ***Key Conclusion 2 (consequence of 1):*** Closing ranks can always be achieved when several vertices with their incident edges are lost by making **connections among the neighbours of the lost vertices**

Closing Ranks

- In two (three) dimensions, *k* degree ve…
  removal requires *k-2 (k-3)* new ed…
- ***Key Conclusion 1:*** Closing ra… always be achieved when one vertex w… incident edges is lost by making **connec… mong neighbours of the lost vertex**
- ***Key Conclusion 2*** … ***equence of 1):*** Closing ranks can alwa… chieved when several vertices wi… incident edges are lost by making … **tions among the neighbours of the lost v… ices**

LOCAL REPAIR PRINCIPLE

Closing Ranks

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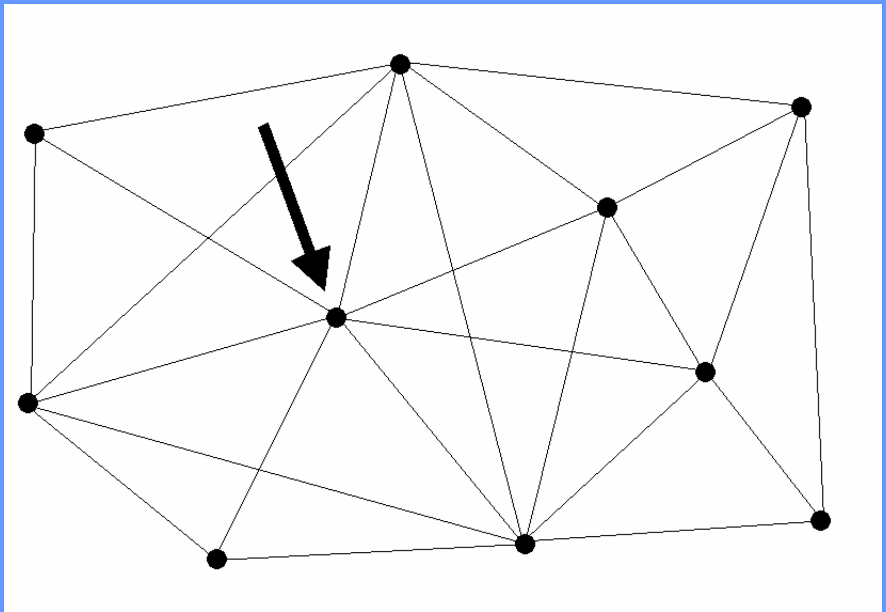
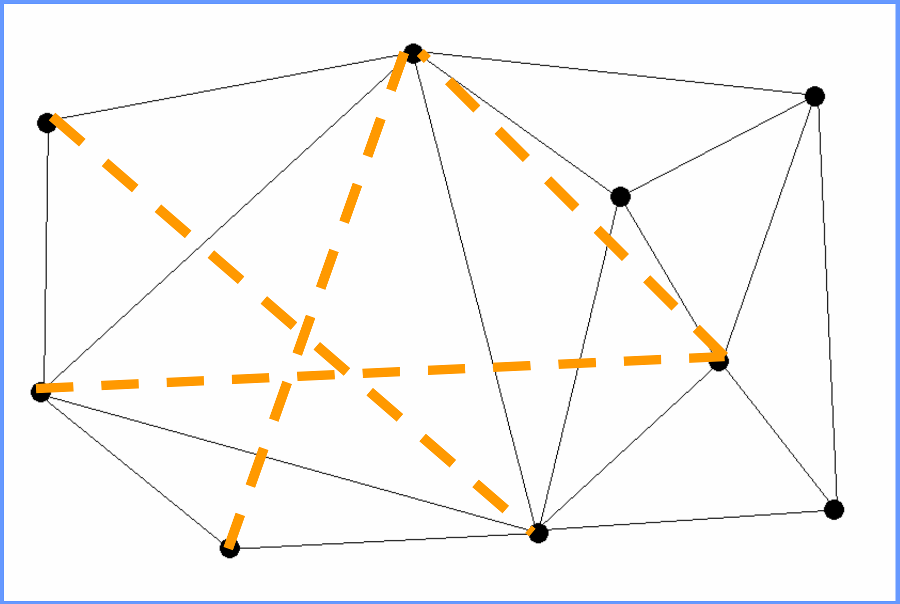
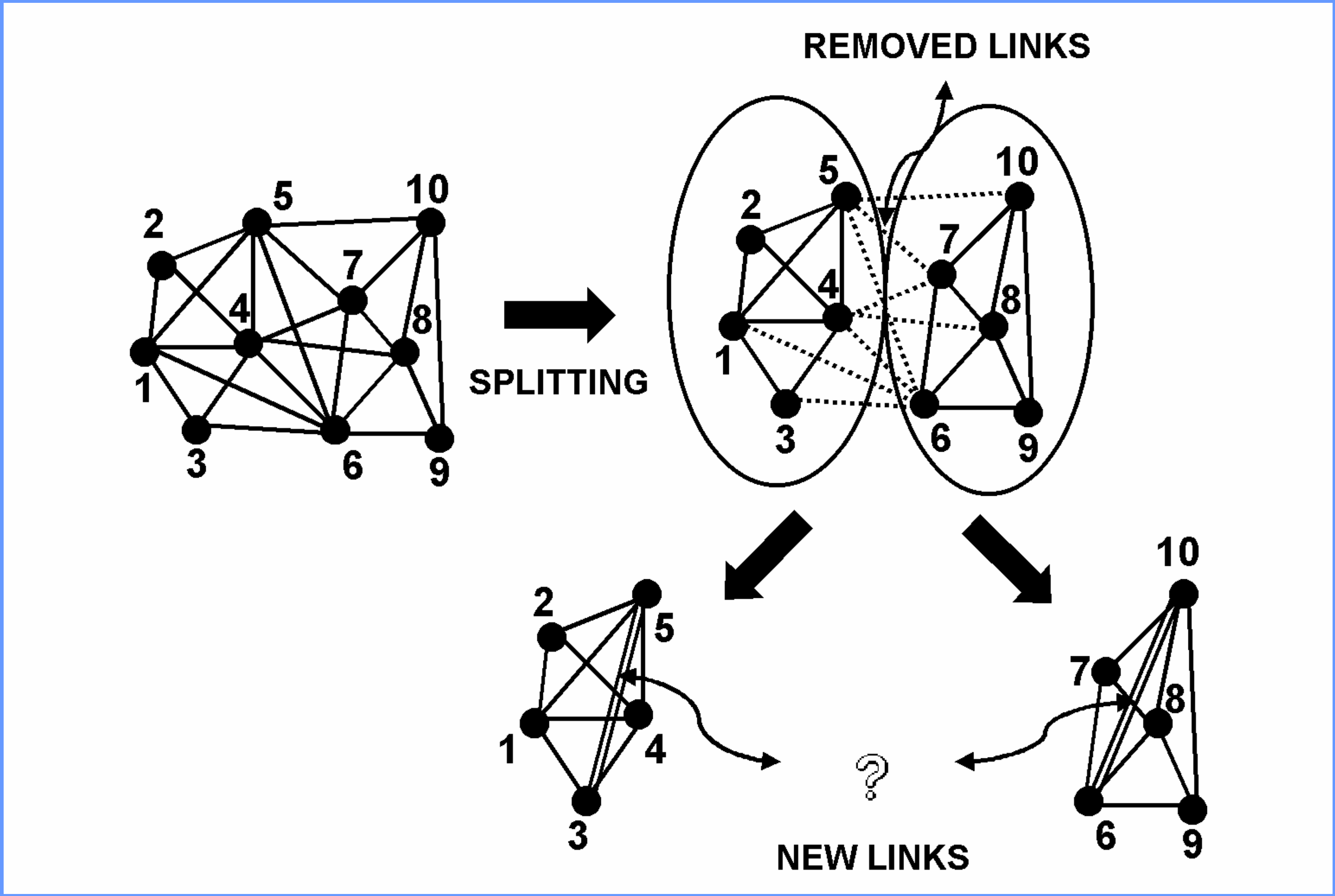
- Remaining links kept and 4 new ones added giving rigidity





Formation Splitting is Closing Ranks!



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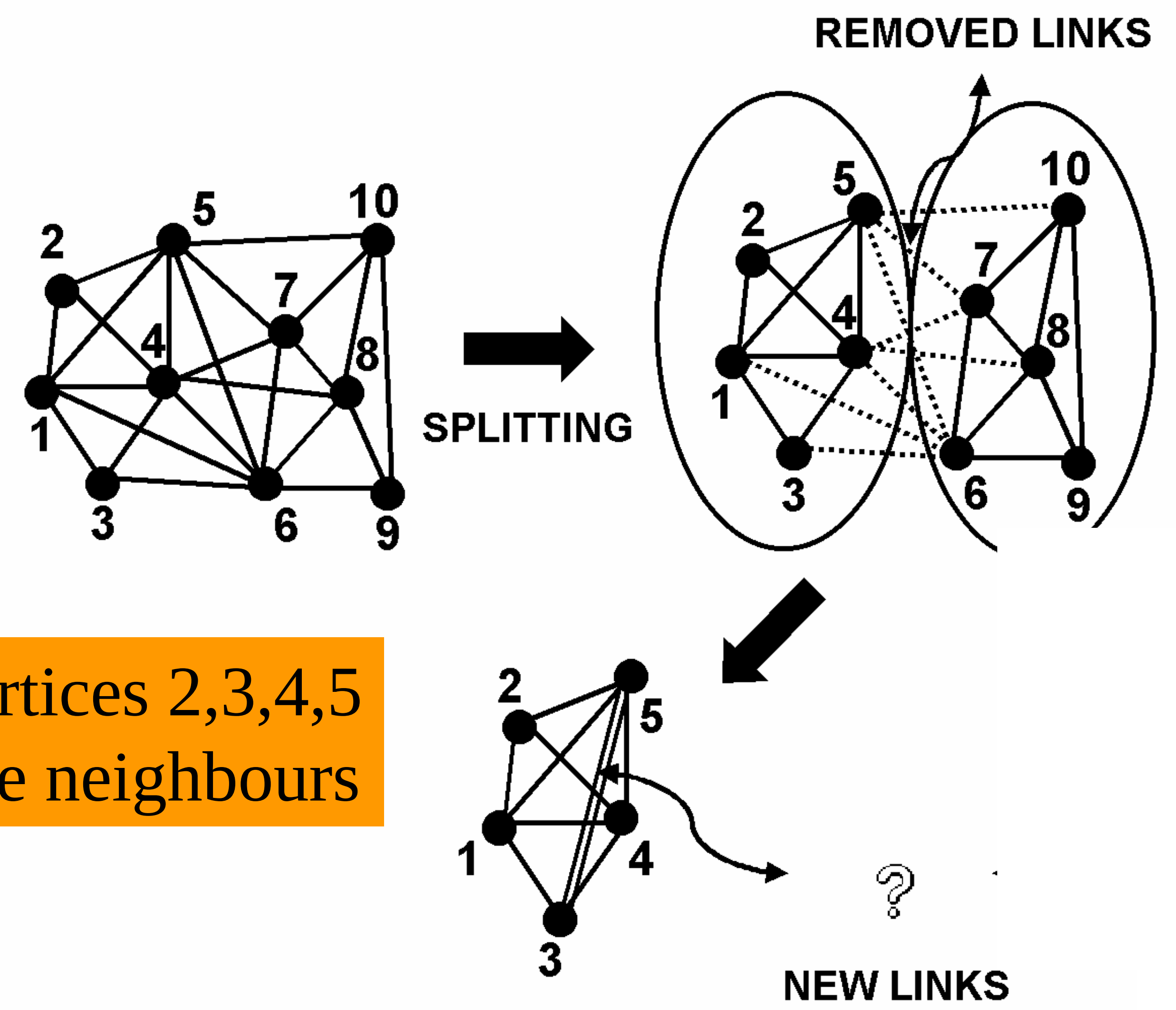

Vertices 2,3,4,5 lose neighbours

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Open problems

- Three dimensional problems;
  - graph theoretic characterization of rigid graphs
  - Henneberg sequences
- Non-minimally (redundantly) rigid problems
- **Efficient** persistence checking
- Dynamic issues of splitting to move round an obstacle
- Dynamic issue of changing formation while preserving rigidity.
- Incorporating directional (bearing) rather than just length constraints to force rigidity