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Analysis of Pilot Symbol Assisted Modulation in Fading Channels Using Finite State Markov Models

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***Abstract*— In this paper, we present an analysis for the channel state estimation and coherent data detection in pilot symbol assisted modulation (PSAM), which is based on the maximum a posteriori (MAP) criterion. The time varying flat fading channel is modelled as the finite state Markov channel (FSMC). The exact channel state estimation error probability is analyzed, using the forward recursion in the Forward-Backward algorithm. The effect of channel state estimation error probability on the receiver symbol error rate (SER) performance in M-ary phase shift keying (PSK) scheme is formulated. The analysis is applied for different channel phase quantization levels and various pilot symbol numbers. The results indicate that BPSK SER performance is saturated with more than 8 to 16 levels of channel phase quantization. For QPSK scheme, 16 levels of channel phase quantization is adequate. Moreover, the transmission of 3 consecutive pilot symbols achieves an acceptable SER performance.**

I. INTRODUCTION

Channel estimation is a fundamental requirement for the coherent detection of phase modulated signals in time varying fading channels. In practical communication receiver designs, training or pilot symbols are periodically inserted between the information symbols to facilitate the channel estimation task. This technique for channel estimation and coherent data detection is often referred to as pilot symbol assisted modulation (PSAM) [1].

Using an information theoretic approach, the authors in [2], [3] have elaborated on the optimum pilot symbol spacing and power allocation in the PSAM transmission, where the continuous-level fading channel is estimated and linear minimum mean squared error (MMSE) of the channel estimation is used to derive the constrained channel capacity or its bounds.

In this paper, we find the optimum quantization resolution for channel estimation and the optimum number of pilot symbols in PSAM transmission. The optimality criterion is to obtain an acceptable receiver symbol error rate (SER) performance, while minimizing the receiver computational complexity. To this end, we model the unknown time varying flat fading channel with a finite state Markov channel (FSMC) [4]. FSMC models are widely used for tracking and estimating the fading channel at the receiver side [5]–[9]. Joint iterative channel estimation and data detection is readily applicable with FSMC modelling of the fading channel [8], [10]. Modelling the fading channel evolution in time with the optimum, finite channel memory order is addressed in [11], [12]. The results indicate that the first-order Markovian assumption is accurately applicable for channel normalized fading rates of $f_D T \lesssim 0.01$, where f_D is the maximum Doppler frequency and T is the symbol period.

In this paper, the exact channel state estimation error probability in the FSMC model is studied, which is based on the maximum a posteriori (MAP) estimation criterion. Calculating the exact channel state estimation error probability is carried out, using the computationally efficient, forward recursive formula in the Forward-Backward or BCJR [13] algorithm.

The effect of channel estimation error probability on the SER in the coherent detection of BPSK and QPSK signalling schemes is analyzed for a wide range of signal to noise ratios (SNR) and channel fading rates. The focus of the analysis is on the channel phase estimation, which poses a greater challenge than the channel amplitude estimation for coherent data detection. The main outcomes of the paper are summarized as follows

1) For BPSK signalling, the receiver SER performance is rapidly saturated with more than 8 to 16 levels of channel phase quantization. This is equivalent to channel phase estimation with only $\pi/4$ to $\pi/8$ phase resolution.
2) For QPSK signalling, estimation of the channel phase with 16 channel states, or equivalently $\pi/8$ phase resolution, is adequate for acceptable receiver SER performance.
3) Transmission of three consecutive pilot symbols results in a reasonable receiver SER performance, for the normalized fading rates $0.001 < f_D T < 0.05$, in a wide range of SNR conditions.

This paper is organized in five sections. Section II reviews the system model. Section III presents the MAP channel state estimation in FSMC models and analyzes the channel phase state estimation error probability and receiver SER performance. In Section IV, numerical results are provided. Section V concludes the paper.

II. SYSTEM MODEL

All signals are represented in their complex, low-pass format. A continuous-time signal $a(t)$ is sampled at the receiver side with the symbol interval T and is presented as $a_n = a(nT)$. Discrete-time sequences will be denoted as vectors, $a_k^n = [a_k, a_{k+1}, ..., a_n]$. Random variables or processes and their realizations are referred to by upper case and corresponding lower case letters, respectively. The probability density function (pdf) of the random variable A is denoted by $f(A = a)$ and is shortened to $f(a)$, where the context is clear.

The channel is assumed to be time-selective, but frequency non-selective (flat) fading, which follows Jakes-Clarke's model [14]. If the fading process is slow enough, so that it is essentially constant over symbol intervals, the sampled received signal at the k^{th} signalling interval is given by

$$r_k = c_k x_k + n_k = \sqrt{\mathcal{E}_s} a_k e^{j\theta_k} e^{j\phi_k} + n_k = \nu_k e^{j\psi_k}, \quad (1)$$

where $x_k = \sqrt{\mathcal{E}_s} e^{j\phi_k}$ is the phase modulated symbol, r_k is the received signal, and n_k is a sample function from the complex, zero-mean, additive white Gaussian noise (AWGN) process, with variance per dimension equal to $\frac{N_0}{2}$. Channel fading gain c_k is a zero-mean stationary complex Gaussian process, with the normalized variance of $\sigma^2 = 0.5$ per dimension. The dynamics of the fading process is determined by its normalized auto-correlation function (ACF) per dimension, given as

$$\rho_n = J_0(2\pi n f_D T) = J_0(2\pi n \frac{v}{\lambda_c} T), \quad (2)$$

where J_0 is the zero-order Bessel function of the first kind, f_D is the maximum Doppler frequency, v is the mobile velocity, and λ_c is the carrier wavelength.

In M-ary PSK signalling, the phase modulated signal, $x_k = \sqrt{\mathcal{E}_s} e^{j\phi_k}$, can take on M different phase values as $\phi_k = \frac{2\pi m}{M}$, $0 \leqslant m \leqslant M - 1$. Following a similar procedure as in [15, pp. 266-268], the conditional polar distribution of the received signal in (1) is written as

$$\begin{aligned} f(\nu_k, \psi_k | a_k, \theta_k, \phi_k) &= \frac{\nu_k}{\pi N_0} \quad (3) \\ &\exp\left[-\frac{\nu_k^2 + \mathcal{E}_s a_k^2 - 2\sqrt{\mathcal{E}_s} a_k \cos(\psi_k - \theta_k - \phi_k)}{N_0}\right]. \end{aligned}$$

In the pilot symbol transmission interval, ϕ_k and as a result, $\omega_k \triangleq \psi_k - \phi_k$ are known. Due to the M-ary PSK symmetry, the distribution of ω_k does not depend on the choice of ϕ_k. Integrating (3) over $0 \leqslant \nu_k < \infty$ will result in the conditional distribution of ω_k, given by

$$f(\omega_k | a_k, \theta_k) = \frac{e^{-\gamma_s a_k^2}}{2\pi} + \frac{\beta_k}{2\sqrt{\pi}} e^{(-\gamma_s a_k^2 + \beta_k^2)} \mathrm{erfc}(-\beta_k), \quad (4)$$

where $\gamma_s \triangleq \frac{\mathcal{E}_s}{N_0}$ and $\beta_k \triangleq \sqrt{\mathcal{E}_s} a_k \cos(\omega_k - \theta_k)$.

III. FSMC MODEL FOR MAP CHANNEL ESTIMATION

In this section, we present FSMC model for MAP estimation of the time varying channel fading phase response. Equation (4) is the distribution of the differential received phase, conditioned on the channel phase θ_k and channel amplitude a_k. By integrating (4) with respect to all possible channel amplitude gains, a model for channel phase-only estimation is obtained as

$$f(\omega_k | \theta_k) = \int_0^\infty f(\omega_k | a_k, \theta_k) f(a_k) da_k, \quad (5)$$

where a_k is assumed to have Rayleigh distribution, given by

$$f(A_k = a_k) = \frac{a_k}{\sigma^2} \exp\left(-\frac{a_k^2}{2\sigma^2}\right), \quad 0 \leq a_k < \infty. \quad (6)$$

As the first step to obtain a FSMC model for the channel phase, the channel phase is quantized into L disjoint regions, with center points given by $q_l = \frac{2\pi l}{L}$, $0 \leqslant l \leqslant L - 1$. The channel is said to be in state $S_k = l$, when the channel phase is in the $q_l - \frac{\pi}{L} \leqslant \theta_k < q_l + \frac{\pi}{L}$ region. Consequently, a quantized version of (5) is obtained for each channel phase state as

$$f(\omega_k | S_k = l) = \frac{L}{2\pi} \int_{q_l - \frac{\pi}{L}}^{q_l + \frac{\pi}{L}} f(\omega_k | \theta_k) d\theta_k. \quad (7)$$

The next step is to quantize the received signal phase difference ω_k into P regions. The quantized version of ω_k is said to be in the region $Z_k = p$, whenever $\frac{2\pi p}{P} \leqslant \omega_k < \frac{2\pi(p+1)}{P}$, $0 \leqslant p \leqslant P - 1$. We have chosen $P \geqslant 2L$, and the pdf in (7) is transformed into the probability mass function of the form

$$\Pr(Z_k = p | S_k = l) = \int_{\frac{2\pi p}{P}}^{\frac{2\pi(p+1)}{P}} f(\omega_k | S_k = l) d\omega_k. \quad (8)$$

Suppose that n pilot symbols are consequently transmitted to the receiver. For simplicity, we assume that the pilot symbols are sent from the signaling interval $k = 1$ to $k = n$. The MAP estimation of the channel phase state, based on the reception of the quantized phase difference z_1^n is equivalent to

$$\widehat{s}_n = \arg \underbrace{\max}_{s_n} \Pr(s_n, z_1^n). \quad (9)$$

Defining $\alpha_n(j) \triangleq \Pr(S_n = j, z_1^n)$, as the forward variable in the Forward-Backward or BCJR algorithm [13], the MAP estimation is rewritten as

$$\widehat{s}_n = \arg \underbrace{\max}_{j} \alpha_n(j). \quad (10)$$

The forward variable is efficiently updated from the time index k to $k + 1$ as

$$\alpha_{k+1}(j) = \left[\sum_{i=0}^{L-1} \alpha_k(i) \mathrm{P}_{ij}\right] \Pr(z_{k+1} | S_{k+1} = j), \quad (11)$$

where P_{ij} is the channel phase state transition probability, given by

$$\begin{aligned} P_{ij} &= \Pr\left(S_{k+1} = j | S_k = i\right) \quad (12) \\ &= \frac{L}{2\pi} \int_{q_i - \frac{\pi}{L}}^{q_i + \frac{\pi}{L}} \int_{q_j - \frac{\pi}{L}}^{q_j + \frac{\pi}{L}} f\left(\theta_{k+1}, \theta_k\right) d\theta_{k+1} d\theta_k, \end{aligned}$$

and defining $\delta \triangleq \rho_1 \cos(\theta_{k+1} - \theta_k)$, $f\left(\theta_{k+1}, \theta_k\right)$ is [16]

$$f(\theta_{k+1}, \theta_k) = \frac{\left(1 - \rho_1^2\right)}{4\pi^2} \frac{\sqrt{(1 - \delta^2)} + \delta\left(\pi - cos^{-1}(\delta)\right)}{\sqrt{\left(1 - \delta^2\right)^3}}. \quad (13)$$

At the n^{th} pilot symbol transmission, the channel state estimation crossover probability is defined to be

$$\begin{aligned} \mathrm{P}_{es}(i \rightarrow j) &= \Pr(\widehat{S}_n = j | S_n = i) \quad (14) \\ &= \sum_{z_1^n : j = \arg\max \alpha_n(.)} \alpha_n(i). \end{aligned}$$

It has to be emphasized that the channel state crossover probability in (14) is exactly calculated for small to moderate values of P and n. At iteration n, $\alpha_n(.)$ has to be calculated for all the P^n possibilities of z_1^n, which exponentially grow with n. Moreover, for large n, α_n will eventually experience underflow [1]. The exact computation of (14) is comfortably possible for $P^n \lessapprox 2^{20}$.

Channel phase state estimates are then used in the data detection mode. Using the state estimation crossover probability in (14), the average SER is written as

$$\mathrm{SER} = \sum_{i=0}^{L-1} \Pr(S_n = i) \sum_{j=0}^{L-1} \mathrm{P}_{es}(i \to j)\mathrm{SER}(i \to j), \quad (15)$$

where $\Pr(S_n = i) = \frac{1}{L}$ for all i, and $\mathrm{SER}(i \to j)$ denotes the symbol error rate when the channel state is estimated to be j, while the actual channel state is i. Whenever the channel phase state is estimated to be j, a phase shift equal to $-\frac{2\pi j}{L}$ is introduced to the received signal, and we obtain

$$\mathrm{SER}(i \to j) = 1 - \int_{-\pi/M - \frac{2\pi j}{L}}^{\pi/M - \frac{2\pi j}{L}} f(\omega_n | S_n = i) d\omega_n, \quad (16)$$

where for simplicity, we have assumed that the information symbol with phase $\phi_n = 0$ was sent. $f(\omega_n|s_n)$ was defined in (7).

IV. NUMERICAL RESULTS

In this section, we study the effect of channel phase quantization resolution and the number of pilot symbols on the symbol error rate (SER) performance of BPSK and QPSK signalling, for various SNR and channel fading rate conditions.

For the numerical analysis, channel state transition probability is determined from (12) and (13). The differential received phase distribution is then derived, using (7) and (8). Assuming that n pilot symbols are transmitted, the forward variable, $\alpha_n(.)$ is determined for all possible z_1^n. Signalling symbol error rate is computed using (14)-(16).

Fig. 1 shows the BPSK SER for $f_D T = 0.001$. The number of channel states varies between $L = 2$ to $L = 16$. As it can be observed from this figure, the SER performance is poor for crude channel phase quantization of $L = 2$ or $L = 4$. However, the gap between the SER performance with 8-level channel phase quantization and 16-level channel phase quantization is negligible, especially with 3 pilot symbols. Using an 8-level phase quantization is equivalent to a $\pi/4$ or 45° phase quantization resolution. This contradicts the expectations, where it is assumed that a very fine channel quantization resolution is required for proper channel estimation. Moreover, with only 3 pilot symbols an acceptable SER performance is achieved.

Fig. 2 shows the BPSK SER for $f_D T = 0.025$. As before, the number of channel phase states is 2, 4, 8, and 16. It can be observed from this figure that the gap between the SER performance with 8-level channel phase quantization and 16-level channel phase quantization is negligible, especially with 3 pilot symbols and high SNR conditions. Even using a 16-level phase quantization scheme is equivalent to a $\pi/8$ or 22.5° phase quantization resolution.

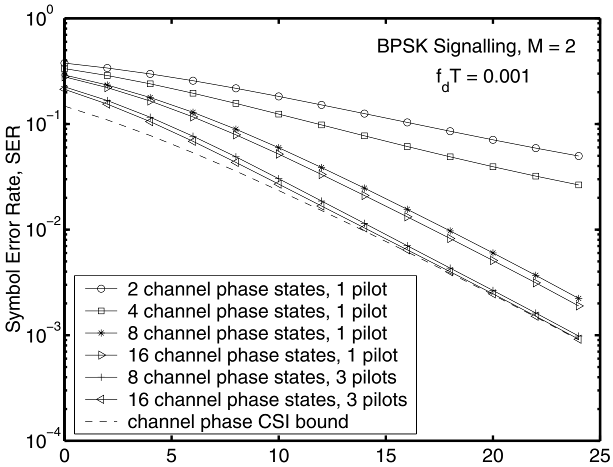

Fig. 1. Symbol error rate performance of BPSK with different pilot symbol numbers and channel phase quantization levels, at $f_D T = 0.001$. 16 channel phase quantization levels and 3 pilot symbols closely achieve the channel phase CSI bound.

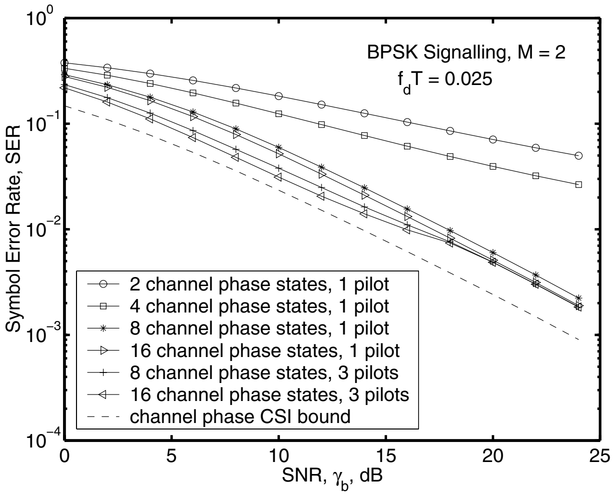

Fig. 2. Symbol error rate performance of BPSK with different pilot symbol numbers and channel phase quantization levels, at $f_D T = 0.025$. SER performance is saturated beyond 16 channel phase quantization levels and 3 pilot symbols.

Next, we compare the SER performance of BPSK and QPSK signalling in time varying fading conditions. For fair comparison, we have kept the SNR per bit ($\gamma_b = \frac{\gamma_s}{\log_2 M}$) equal for the two signalling schemes. However, keeping the transmission power the same for both BPSK and QPSK, the symbol period for QPSK will be twice as long as BPSK symbol period. Consequently, QPSK will experience double the fading rate as BPSK.

Fig. 3 shows the results for the BPSK at fading rate of $f_D T = 0.001$ and QPSK at fading rate of $f_D T = 0.002$, with 3 pilot symbols. The symbol error rate of QPSK with

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[1] Please, note that unlike turbo decoding algorithm, no scaling is allowed here, and the unscaled values of $\alpha_k(i)$ are required in (14).

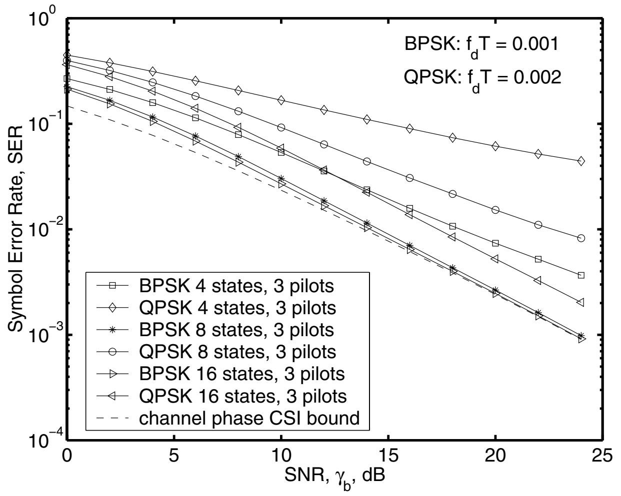

Fig. 3. Comparison of BPSK at $f_DT = 0.001$, and QPSK SER at $f_DT = 0.002$. 16 levels of channel phase quantization and 3 pilot symbols is adequate for both BPSK and QPSK schemes.

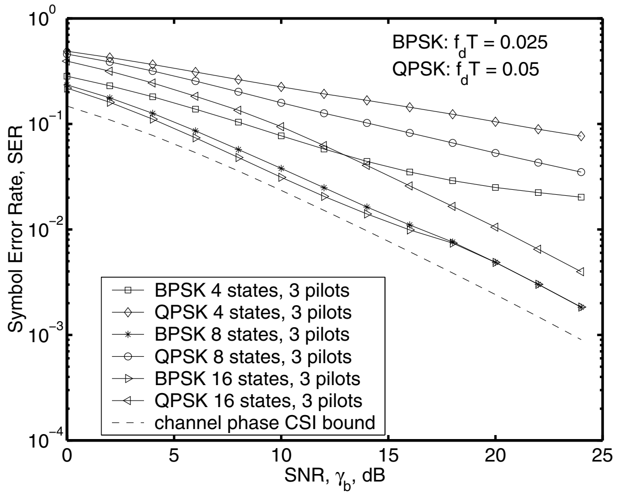

Fig. 4. Comparison of BPSK at $f_DT = 0.025$, and QPSK SER at $f_DT = 0.05$. QPSK is undergoing faster fading conditions than BPSK. As a result, QPSK SER is more than twice the BPSK SER, at low SNR region.

16 channel phase quantization levels is about twice the BPSK SER with 8-16 channel phase states, which is as expected [15, p. 269]. Since 8-16 channel phase quantization levels for BPSK signalling almost achieves its SER performance bound, we conclude that $2\pi/16$ phase quantization resolution is adequate for QPSK signalling in $f_DT = 0.002$.

Fig. 4 shows the results for the BPSK at fading rate of $f_DT = 0.025$ and QPSK at fading rate of $f_DT = 0.05$, with 3 pilot symbols. In the high SNR region, the symbol error rate of QPSK with 16 channel phase quantization levels is about twice the BPSK SER with 8-16 channel phase states. For lower SNR conditions, the QPSK SER is about 2.5-3 times more than BPSK SER. This is due to the fact that QPSK is experiencing a faster fading condition than BPSK, which results in more symbols being mixed up.

V. CONCLUSIONS

We studied the effect of channel phase quantization and the number of pilot symbols on the channel state estimation error probability and M-ary PSK symbol error rate. The flat fading channel was modelled as the finite state Markov channel and the exact channel state estimation error probability was investigated, using the forward recursion in the BCJR algorithm. It was demonstrated through numerical analysis that the SER performance of BPSK and QPSK signalling is saturated for more than 16 levels of channel phase quantization, which is 4 to 8 times the number of signalling constellation points, M. Furthermore, the SER performance does not improve much beyond 3 pilot symbols, over a wide range of SNR and fading rate conditions.

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