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Finite-State Markov Modelling of Frequency-Selective Fading Channels with Correlated Taps

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***Abstract*—We consider the problem of modelling a randomly time-varying frequency-selective fading channel as a finite-state Markov channel (FSMC). For a fading channel with two correlated taps and given statistical parameters, the accuracy of an FSMC is assessed by comparing its information rate, when the receiver has ideal channel-state information, to that of the original continuous-valued channel. We show that in order to construct an FSMC with a given desired accuracy, fewer states are required if the correlation between taps is taken into account than if the taps are treated as being independent. These results demonstrate the suitability and accuracy of FSMCs for modelling a time-varying frequency-selective fading channel with memory.**

I. Introduction

A. Motivation and background

1) The finite-state Markov channel: The classical finite-state Markov channel [1] has been widely used for modelling time-varying wireless fading channels in a variety of applications. Such applications include 1) modelling of channel error-bursts and performance evaluation during system design stages, 2) joint channel-estimation and decoding implemented at the receiver, and 3) adaptive coding and power-allocation implemented at the transmitter. For a survey of existing applications in the literature, refer to [2]. Two main reasons for this success in applying FSMC models to fading channels is the model's versatility, in being able to represent a wide range of time-varying fading-channel conditions, and its mathematical tractability. For example, due to the finite-ness and Markovian property of FSMC states, joint iterative channel-estimation and decoding is possible via the maximum *a posteriori* (MAP) algorithm [3].

Until now in the literature, FSMC models have been considered for *frequency-flat* time-varying fading channels, where the transmitted signal is affected by a single random and time-varying fading-tap. In this case, FSMC states are often obtained by partitioning or quantising the range of values of the fading-tap into a number of non-overlapping intervals. In many applications of interest, however, the multipath fading channel exhibits both *frequency-selectivity* and time-variations. As a result, the transmitted signal is affected by multiple taps that change over time in a random manner. One might argue that a frequency-selective fading channel is a superposition of individual taps and proceed to partition each tap *separately* into a number of substates and then combine these substates to form the overall FSMC model.

However, there is at least one immediate drawback to the separate-taps FSMC modelling approach: the overall number of FSMC states grows rapidly as the memory-length of the frequency-selective channel increases. This, in turn, can render the use of an FSMC model computationally intensive, especially for receiver implementation. Motivated by this problem, we consider *combined* modelling of all taps. The significance of this joint approach is made clearer by noting that even when the physical multipath channel possesses the property of uncorrelated scattering (US) [4], fading-taps in the low-pass received-signal model become correlated due to matched filtering (MF) at the receiver [5]. By taking these correlations into account and performing vector-quantisation (VQ) of fading-taps, one expects to be able to reduce the number of FSMC states that is required for satisfactory representation of the original frequency-selective fading channel.

B. Contributions

This paper represents the first step towards systematic combined FSMC modelling of frequency-selective fading channels. We derive the parameters of FSMC models for a general frequency-selective fading channel and explain how a simulated fading-sequence can be used in vector-quantisation of the fading-process and in the computation of FSMC parameters.

We compare the information rates of the obtained FSMCs with the information rate of the original continuous-state fading channel under the assumption of ideal channel-state information (CSI) at the receiver. The closeness of an FSMC's information rate to that of the original channel is one measure of the model's accuracy and applicability. The CSI assumption is valid where the rate of channel time-variation is relatively low ($f_D T_s \lesssim 0.01$). For faster-fading cases, one may need to consider FSMC information rates with no CSI available at the receiver. In any case, the information rate analysis provides an indication of rates that can be obtained through FSMC modelling of a continuous-state channel, and information rate is a suitable criterion by which to assess application of FSMCs at the receiver for channel-estimation and equalisation. It also provides a measure of the number of states needed for a close approximation of the original channel.

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The remainder of the paper is organised as follows. Section II details our chosen model for the fading channel, explains how the correlation between fading-gains arises, and shows how the fading can be modelled as a finite-order autoregressive (AR) process. In Section III the parameters of an FSMC model with a given number of states are derived, using a first-order AR simulation of the fading-process. In Section IV we discuss the use of an FSMC's ideal-CSI information rate as a criterion for assessing the model's accuracy. In Section V we provide and compare numerical results obtained for both the combined-taps and the independent-taps FSMCs. We conclude in Section VI.

To allow clearer analysis and exposition of ideas, we present the main results of the paper for an ISI fading channel with two taps. However, most of the observations and results in Sections II and III are applicable generally to a channel with an arbitrary number P of taps.

II. Model of the Frequency-Selective Fading Channel with Matched Filter at the Receiver

A. Observation equation

In the fading channel with P-tap intersymbol interference (ISI), the complex-valued low-pass received signal y_ℓ at the ℓ-th symbol-time, given transmitted inputs $\ldots, x_{\ell-1}, x_\ell$ and additive white noise z_ℓ, is

$$y_\ell = \sum_{p=0}^{P-1} h_{p,\ell} x_{\ell-p} + z_\ell, \tag{1}$$

where the fading coefficient at tap p and symbol-time ℓ is

$$h_{p,\ell} \equiv a_{p,\ell} e^{i\theta_{p,\ell}},\ p = 0, 1, \ldots, P-1, \text{ and } z_\ell \sim N_{\mathbb{C}}(0, N_0).$$

The fading $h_{p,\ell}$ at each tap p has a wide-sense-stationary zero-mean circularly symmetric complex-Gaussian distribution, and therefore the fading-amplitude $a_{p,\ell}$ has a Rayleigh distribution with probability density function

$$f_{A_p}(a_p) = \frac{a_p}{\sigma_p^2} \exp\left(-\frac{a_p^2}{2\sigma_p^2}\right),$$

where σ_p^2 is the variance of each of the real and imaginary parts of $h_{p,\ell}$, and stationarity has allowed us to drop the time-index ℓ. Furthermore, the fading-phase $\theta_{p,\ell}$ is uniformly distributed between 0 and 2π. The expected fading-power $E[|h_{p,\ell}|^2]$ can vary from tap to tap, although the assumption of wide-sense stationarity implies that it does not vary with time at a given tap. We assume that for each time-step ℓ the fading-phases $\theta_{p,\ell}$ have been tracked and are known at the receiver; however the amplitudes $a_{p,\ell}$ are not necessarily known.

With phase knowledge at the receiver, the observation equation (1) can be written in such a way that it depends on only $P-1$ phase-differences rather than P phases. For example, with $P = 2$ taps the observation equation becomes

$$y'_\ell = a_{0,\ell} x_\ell + a_{1,\ell} \exp(i\phi_\ell) x_{\ell-1} + z'_\ell \tag{2}$$

where $y'_\ell \equiv \exp(-i\theta_{0,\ell}) y_\ell$, $z'_\ell \equiv \exp(-i\theta_{0,\ell}) z_\ell$, and $\phi_\ell \equiv (\theta_{1,\ell} - \theta_{0,\ell})$ modulo 2π. Note that $z'_\ell \sim N_{\mathbb{C}}(0, N_0)$. This two-tap example will be used again in Sections III, IV, and V, where for clarity we drop the $'$ superscript on y_ℓ and z_ℓ.

The inputs $X_0, X_1, \ldots$ are i.i.d. BPSK symbols with normalised power, so that $X_\ell \in \{-1, 1\} \equiv \mathcal{X}$ for each ℓ. The additive white noise z_ℓ has a circularly symmetric complex-Gaussian distribution; therefore the real and imaginary parts of z_ℓ are independent, and each has a Gaussian distribution with mean zero and variance $N_0/2$.

B. The origin of correlation in fading-taps

Let $h_{p,\ell}$ be the fading at tap p and time-step ℓ. Even when the multipath fading possesses the property of uncorrelated scattering, correlation between fading at different taps can arise through the action of a matched filter (commonly used in practical receivers), which we now describe [5]. The sequence of inputs x_k is modulated at the transmitter using a pulse-shaping function; scattering by the wireless channel results in several copies of the modulated input arriving at the receiver, each copy undergoing time-dependent delay, attenuation, and shift in phase; a matched filter is used for demodulation, and the resulting signal is sampled at discrete time-intervals.

The modulated signal is

$$s(t) = \sum_{k=0}^{B-1} x_k P(t - kT_s),$$

where T_s is the symbol period, there are B inputs per block of data, and $P(t)$ is a pulse-shaping function that is non-zero only for $t \in [0, T_s]$.

The channel response $g(\tau)$, assuming a fixed number J of fixed delays τ_j that satisfy $\tau_j \geq 0$ and $j < j' \Leftrightarrow \tau_j < \tau_{j'}$, is

$$g(t, \tau) = \sum_{j=0}^{J-1} g_j(t) \delta(\tau - \tau_j),$$

where $g_j(t)$ is the time-varying complex-valued gain at the j-th *physical tap*. Each amplitude $|g_j|$ has a Rayleigh distribution with variance $\sigma_g^2(j)$. Assuming wide-sense-stationary uncorrelated scattering (WSSUS) [4], the covariance between gains is

$$E[g_j(t)\overline{g_{j'}(t-v)}] = \delta(j - j')\sigma_g^2(j) r_g(v), \tag{3}$$

where $r_g(\cdot)$ is the autocorrelation in time of the gain g_j and is the same for all j; by Clarke's model for each independent gain, we have

$$r_g(v) = J_0(2\pi f_D v)$$

with f_D the maximum Doppler frequency-shift and $J_0(\cdot)$ the zeroth-order Bessel function of the first kind.

We ignore the effect of AWGN in the channel, as it has no effect on the correlations between fading at the taps. The received signal, then, is

$$\begin{aligned} u(t) &= s(t)\star g(t,\tau) = \sum_{j=0}^{J-1} g_j(t)s(t-\tau_j) \\ &= \sum_{k=0}^{B-1} x_k \sum_{j=0}^{J-1} g_j(t)P(t-\tau_j-kT_s). \end{aligned}$$

The matched filter is defined to be $f(t)\equiv\overline{P}(-t)$, and hence, noting that $f(\cdot)$ is non-zero only on $[-T_s,0]$, the demodulated signal is

$$w(t) = u(t)\star f(t) = \int_t^{t+T_s} u(\alpha)f(t-\alpha)d\alpha.$$

The signal sampled at the ℓ-th time-step is

$$\begin{aligned} w_\ell \equiv w(\ell T_s) &= \sum_k x_k \int_{\ell T_s}^{(\ell+1)T_s} \sum_{j=0}^{J-1} g_j(\alpha)\times \\ &\quad P(\alpha-\tau_j-kT_s)f(\ell T_s-\alpha)d\alpha. \end{aligned}$$

We assume each gain $g_j(t)$ varies sufficiently slowly with time that $g_j(\alpha)$ can be considered a constant $g_j(\ell T_s)$ during the ℓ-th symbol-period; then

$$w_\ell = \sum_k x_k \sum_{j=0}^{J-1} g_j(\ell T_s)\int P(\alpha-\tau_j-kT_s)\overline{P}(\alpha-\ell T_s)d\alpha.$$

Letting $\beta\equiv\alpha-\ell T_s$ and

$$P_{\text{total}}(t)\equiv\int_0^{T_s} P(\beta+t)\overline{P}(\beta)d\beta,$$

we obtain

$$\begin{aligned} w_\ell &= \sum_k x_k \sum_{j=0}^{J-1} g_j(\ell T_s)P_{\text{total}}((\ell-k)T_s-\tau_j) \\ &\equiv \sum_k x_k h_{\ell-k}(\ell). \end{aligned}$$

Making the substitution $p=\ell-k$, and observing that the properties of the pulse-shaping function imply h_p is zero for $p<0$, gives

$$w_\ell = \sum_{p=0}^{P-1} x_{\ell-p}h_{p,\ell}$$

where

$$h_{p,\ell}\equiv\sum_{j=0}^{J-1} g_j(\ell T_s)P_{\text{total}}(pT_s-\tau_j) \tag{4}$$

is the (complex-valued) fading at tap p at the ℓ-th time-step. Note that $P_{\text{total}}(t)$ is non-zero only for $t\in(-T_s,T_s)$, so that when summing over $P_{\text{total}}(pT_s-\tau_j)$, for a given j we need consider only the values $p=\lfloor\tau_j/T_s\rfloor,\ldots,\lceil\tau_j/T_s\rceil$. Hence $h_{p,\ell}$ is non-zero only for $p\in(\lfloor\tau_0/T_s\rfloor,\ldots,\lceil\tau_{J-1}/T_s\rceil)$. Thus the number P of taps depends on the values of the delays τ_j, and specifically

$$P=\left\lceil\frac{\tau_{J-1}}{T_s}\right\rceil-\left\lfloor\frac{\tau_0}{T_s}\right\rfloor+1.$$

We define the fading-vector

$$\mathbf{h}_\ell\equiv(h_{0,\ell}\ h_{1,\ell}\ \cdots\ h_{P-1,\ell})^t,$$

the delay-vector

$$\mathbf{g}_\ell\equiv(g_0(\ell T_s)\ g_1(\ell T_s)\ \cdots\ g_{J-1}(\ell T_s))^t,$$

and the P-by-J matrix $\mathbf{P}$ with entries $P_{a,b}\equiv P_{\text{total}}((a-1)T_s-\tau_{b-1})$, and observe that (4) can be written

$$\mathbf{h}_\ell=\mathbf{P}\,\mathbf{g}_\ell.$$

The mth-lag covariance matrix is

$$\begin{aligned} \mathbf{R_{hh}}(m) &\equiv E[\mathbf{h}_\ell\mathbf{h}_{\ell-m}^H] \\ &= \mathbf{P}E[\mathbf{g}_\ell\mathbf{g}_{\ell-m}^H]\mathbf{P}^H. \end{aligned}$$

But by (3), upon defining $\mathbf{D}\equiv\text{diag}(\ \sigma_g^2(0),\ \sigma_g^2(1),\ \ldots,\ \sigma_g^2(J-1)\ )$, we have

$$\mathbf{R_{hh}}(m)=r_g(mT_s)\mathbf{PDP}^H. \tag{5}$$

The matched filter thus has the effect of introducing correlations between the erstwhile independent taps.

C. Simulation of fading as an autoregressive process

We show how to use an AR model to generate a sequence of simulated fading variates, whose statistics up to a chosen finite order are identical to those of the modelled fading-process. AR modelling is amenable to mathematical analysis, and a given model's parameters can be easily re-derived in the event of changed channel-conditions; furthermore, by choosing a sufficiently high model-order, an AR model can be found that is arbitrarily close to Clarke's model [6]. AR modelling has been previously applied to fading channels in work such as [7].

1) State-space equation of the AR process: Fading at correlated taps is simulated by assuming that the fading-vector

$$\mathbf{h}_\ell\equiv(h_{0,\ell}\ h_{1,\ell}\ \cdots\ h_{P-1,\ell})^t$$

conforms to an AR model of finite order q,

$$\mathbf{h}_\ell=-\sum_{k=1}^{q}\mathbf{A}_k\mathbf{h}_{\ell-k}+\mathbf{w}_\ell,$$

where $\mathbf{w}_\ell$ is a complex-valued vector of white Gaussian process-noise with covariance matrix $\mathbf{Q}=E[\mathbf{w}_\ell\mathbf{w}_\ell^H]$.

2) Levinson-Wiggins-Robinson algorithm: The covariance matrices $\mathbf{R_{hh}}(m)$ in (5) above are used to determine the optimal, in a minimum mean-squared-error (MMSE) sense, AR filter coefficients $\mathbf{A}_k$ and covariance matrix $\mathbf{Q}$, via the multivariate Yule-Walker equations

$$\mathbf{R_{hh}}(m) = \begin{cases} -\sum_{k=1}^{q}\mathbf{R_{hh}}(m-k)\mathbf{A}_k^H, & 1\le m\le q; \\ -\sum_{k=1}^{q}\mathbf{R_{hh}}(-k)\mathbf{A}_k^H+\mathbf{Q}, & m=0. \end{cases}$$

The multivariate Yule-Walker equations are solved efficiently using the Levinson-Wiggins-Robinson (LWR) algorithm [8]. The real and imaginary components of the fading h_p at each tap p must be independent [9] and have means of zero in order for the fading to be Rayleigh. The Cholesky decomposition is used to obtain the factorisation $Q=GG^H$, and the process-noise is then generated via $\mathbf{w}_\ell = G\mathbf{z}_l$ where $\mathbf{z}_l$ is a $P \times 1$ vector of independent complex-Gaussian variates with unit variance and mean zero.

III. Deriving the Parameters of an FSMC

Recall that for each tap p and symbol-time index ℓ, we define $a_{p,\ell} \equiv |h_{p,\ell}|$; now let $\mathbf{a}_\ell \equiv (a_{0,\ell}, a_{1,\ell}, \ldots, a_{P-1,\ell})$ be the vector of fading-amplitudes at the P taps at time-index ℓ. The tap-phases $\theta_{p,\ell}$ at time-index ℓ are assumed to be known at the receiver at that time.

Given the domains of the channel-input process X and the channel-output process Y, the parameters of an FSMC are its non-observable Markov chain (comprising its finitely many states and their state-transition probabilities) and its channel observation law (comprising the conditional probability of the channel-output given the channel-input and state) [1].

We shall obtain the FSMC's parameters from the statistics of a first-order simulation of the fading-process.

Note that the channel-output is quantised in the following way. The real part Y_R of the channel-output Y is quantised into M_R distinct intervals $D_{R,m_R} \equiv [w_{R,m_R}, w_{R,m_R+1})$, where $m_R \in \{0, 1, \ldots, M_R - 1\}$, $w_{R,0} = -\infty$, and $w_{R,M_R} = \infty$. Similarly the imaginary part Y_I of Y is quantised into M_I intervals $D_{I,m_I} \equiv [w_{I,m_I}, w_{I,m_I+1})$. Hence Y is partitioned into $M_R M_I$ discrete regions $D_{m_R,m_I} \equiv D_{R,m_R} + iD_{I,m_I}$. (In the numerical analysis, we shall choose $M_R = M_I = 2$.)

A. *Definition of the finite-valued FSMC state*

The states of an FSMC of the frequency-selective fading channel are *composite states*; each comprises a *channel-state* (representing fading at P taps) and an *ISI state* (representing the previous $P-1$ inputs), and these evolve independently.

Consider the transformed version (2) of the observation equation. In order to define a finite number of channel-states, the continuous domains $[0, \infty)^P$ and $[-\pi, \pi)^{P-1}$ of $\mathbf{a}$ and of the phase-differences $(\phi_1, \phi_2, \ldots, \phi_{P-1})$, respectively, must be quantised. Ideally the continuous-valued phase-differences are known exactly at each time ℓ, but in order to construct a finite-state Markov model and to efficiently compute its parameters, each phase-difference must be partitioned into a number of non-overlapping intervals. Choosing a sufficiently large number of intervals will ensure that the error inherent in quantisation is negligible.

Consider the case with $P = 2$ taps. The domain of $\mathbf{a}$ is partitioned into a finite number K of regions R_k, where $k \in \{1, 2, \ldots, K\}$, as in Figure 1 where $K = 21$. For simplicity, the domain $[-\pi, \pi)$ of ϕ is independently partitioned into M equal-width intervals $\{T_v : v = 1, 2, \ldots, M\}$. For each symbol-time index ℓ, let $L_\ell = (U_\ell, V_\ell)$ be the finite-valued channel-state at stage ℓ, where

$$U_\ell = k \Leftrightarrow \mathbf{a}_\ell \in R_k$$

and

$$V_\ell = v \Leftrightarrow \phi_\ell \in T_v.$$

We index the realisations of the state L using

$$L_\ell = k_L \Leftrightarrow U_\ell = k, V_\ell = v, \text{ and } k_L = (k-1)M + v. \quad (6)$$

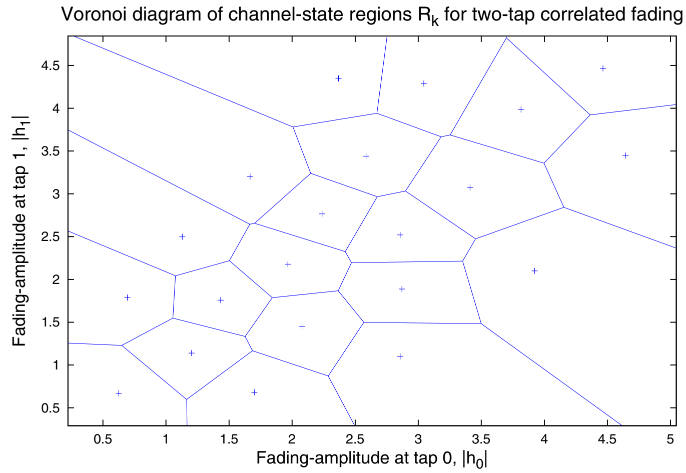

Fig. 1. Voronoi diagram of centroids of regions R_k ($K = 21$ regions)

The amplitudes-states U of the FSMC are defined through vector-quantisation, using an accelerated version of the k-means algorithm [10] to construct K regions R_k that partition the continuous-valued space of vectors $\mathbf{a}$. The k-means algorithm belongs to the EM (expectation-maximisation) family of algorithms, and has been proven to converge to a locally optimal solution [11]. The k-means algorithm is applied to a long simulated sequence of vectors $\mathbf{a}$, which approximates the most likely vectors in the space of amplitudes.

We define the *steady-state* (stationary) probability of a channel-state to be

$$\pi^L_{k,v} \equiv \lim_{\ell\to\infty} \Pr(L_\ell = (k, v)) = \lim_{\ell\to\infty} \Pr(\mathbf{a}_\ell \in R_k \text{ and } \phi_\ell \in T_v).$$

This is estimated by processing the long sequence $((\mathbf{a}_n, \phi_n) : 1 \le n \le N)$ of simulated fading-amplitudes $\mathbf{a}$ and phase-differences ϕ. The above application of the k-means algorithm yields a sequence (u_n) of the particular centroids assigned to individual vectors $\mathbf{a}_n$ in the simulated fading-sequence; it is also trivial to assign intervals (v_n) to the simulated phase-differences (ϕ_n). The estimated steady-channel-state probability is

$$\pi^L_{k,v} \propto |\{(u_n, v_n) = (k, v) : 1 \le n \le N\}|;$$

these probabilities are normalised so that they sum to 1.

Finally, let $Q_\ell \equiv X_{\ell-1}$ be the ISI state, where $X_{\ell-1} \in \mathcal{X}$, and let $S_\ell \equiv (Q_\ell, L_\ell)$ be the composite state of the FSMC.

B. *State-transition probabilities*

The matrix $\mathbf{P}_L$ of channel-state transition-probabilities is defined by

$$(\mathbf{P}_L)_{k_L,k'_L} \equiv \Pr(L_\ell = k'_L \mid L_{\ell-1} = k_L);$$

for $P = 2$ taps we estimate these probabilities as above, by processing the sequence $((u_n, v_n) : 1 \le n \le N)$ of simulated variates discretised into their respective regions and intervals.

Setting $k_L = (k-1)M + v$ and $k'_L = (k'-1)M + v'$ as in (6), we have

$$(\mathbf{P}_L)_{k_L, k'_L} = \Pr(U_\ell = k', V_\ell = v' \mid U_{\ell-1} = k, V_{\ell-1} = v),$$

for which the obtained estimate is

$$\frac{|\{(u_{n-1}, v_{n-1}, u_n, v_n) = (k, v, k', v') : 1 \le n \le N-1\}|}{|\{(u_{n-1}, v_{n-1}) = (k, v) : 1 \le n \le N-1\}|}.$$

C. Channel observation law

For clarity, we give the channel observation law for the special case in which there are $P = 2$ taps. The channel observation law is $\Pr(Y_\ell | X_\ell, S_\ell) = \Pr(Y_\ell | X_\ell, Q_\ell, U_\ell, V_\ell) = \Pr(Y_\ell | U_\ell, V_\ell, X_{\ell=1}^\ell)$.

But $\Pr(Y_\ell \in D_{m_R, m_I} \mid U_\ell = k, V_\ell = v, X_{\ell-1}^\ell = x_{\ell-1}^\ell) =$

$$\int f(\mathbf{a}, \phi, Y_\ell \in D_{m_R, m_I} | \mathbf{A}_\ell \in R_k, \Phi_\ell \in T_v, x_{\ell-1}^\ell)\, d\phi\, d\mathbf{a}$$

$$\begin{aligned} &\propto \int_{\mathbf{a} \in R_k, \phi \in T_v} f(\mathbf{a}, \phi) \Pr(Y_\ell \in D_{m_R, m_I} | \mathbf{a}, \phi, x_{\ell-1}^\ell)\, d\phi\, d\mathbf{a} \\ &= E_{\mathbf{a} \in R_k, \phi \in T_v} [\Pr(Y_\ell \in D_{m_R, m_I} | \mathbf{a}, \phi, x_{\ell-1}^\ell)], \end{aligned} \tag{7}$$

where $\Pr(Y_\ell \in D_{m_R, m_I} | \mathbf{A}_\ell = \mathbf{a}, \Phi_\ell = \phi, X_{\ell-1}^\ell = x_{\ell-1}^\ell) =$

$\Pr(Y_{R,\ell} \in D_{R, m_R} | \mathbf{a}, \phi, x_{\ell-1}^\ell) \times \Pr(Y_{I,\ell} \in D_{I, m_I} | \mathbf{a}, \phi, x_{\ell-1}^\ell)$.

Note that $\Pr(Y_\ell \in D_{R, m_R} | \mathbf{a}, \phi, x_{\ell-1}^\ell) =$

$$\begin{aligned} &Q\left(\frac{w_{R, m_R} - a_0 x_\ell - a_1 x_{\ell-1} \cos \phi_\ell}{\sqrt{N_0/2}}\right) - \\ &Q\left(\frac{w_{R, m_R+1} - a_0 x_\ell - a_1 x_{\ell-1} \cos \phi_\ell}{\sqrt{N_0/2}}\right), \end{aligned} \tag{8}$$

where $Q(\cdot)$ is equal to one minus the cumulative distribution function of the standard Gaussian random variable; a similar expression exists for $\Pr(Y_\ell \in D_{I, m_I} | \mathbf{a}, \phi, x_{\ell-1}^\ell)$.

Computing the channel law requires evaluation of the triple integral (7), which is computationally prohibitive even when there is a known closed form for the joint p.d.f. $f(\mathbf{a}, \phi)$. Hence the integral is approximated numerically using a Monte Carlo technique that sums over the long simulated sequence $((\mathbf{a}_n, \phi_n) : 1 \le n \le N)$. Thus the integral is proportional to

$$\sum_{\mathbf{a}_n \in R_k, \phi_n \in T_v} \Pr(Y_n \in D_{m_R, m_I} | \mathbf{a}_n, \phi_n, x_{n-1}^n)$$

and the channel-law probabilities are normalised so that

$$\sum_{m_R, m_I} \Pr(Y_\ell \in D_{m_R, m_I} | U_\ell = k, V_\ell = v, x_{\ell-1}^\ell) = 1.$$

IV. Evaluating the accuracy of an FSMC

For a fading channel with given statistical parameters, the accuracy of an FSMC with a fixed number of states is assessed by computing its information rate under the assumption of ideal CSI; the closer the rate is to that of the original continuous-valued channel, the more accurate the model is judged to be.

For the channel with $P = 2$ taps, the ideal CSI is

$$C_\ell \equiv (\mathbf{A}_\ell = (A_{0,\ell}, A_{1,\ell}), \Phi_\ell = \phi_\ell, X_{\ell-1}),$$

and it can be shown that for arbitrary P, the information rate $I(\mathcal{Y}; \mathcal{X} | \mathcal{C})$ between the input process $\mathcal{X}$ and the output process $\mathcal{Y}$, given the ideal-CSI process $\mathcal{C}$, satisfies

$$I(\mathcal{Y}; \mathcal{X} | \mathcal{C}) = \lim_{N \to \infty} \frac{1}{N} \sum_{\ell=1}^N I(Y_\ell; X_\ell | C_\ell)$$

where owing to stationarity of the fading-process,

$$I(Y_\ell; X_\ell | C_\ell) \equiv H(Y_\ell | C_\ell) - H(Y_\ell | X_\ell, C_\ell)$$

does not depend upon ℓ. Hence

$$I(\mathcal{Y}; \mathcal{X} | \mathcal{C}) = I(Y_\ell; X_\ell | C_\ell)$$

for any ℓ.

A. Information rate of the continuous-valued channel

For the channel with continuous-valued fading-amplitude, we employ the conditional probability (8) to compute $H(Y_\ell | C_\ell)$

$$\begin{aligned} &= \int_{[0,\infty)^2 \times [-\pi, \pi)} f(\mathbf{A}_\ell = \mathbf{a}, \Phi_\ell = \phi) \sum_{x_{\ell-1} \in \mathcal{X}} \Pr(x_{\ell-1}) \times \\ &\quad H(Y_\ell | \mathbf{A}_\ell = \mathbf{a}, \Phi_\ell = \phi, X_{\ell-1} = x_{\ell-1})\, d\phi\, d\mathbf{a} \\ &= \int f(\mathbf{a}, \phi) H(Y_\ell | \mathbf{a}, \phi, X_{\ell-1})\, d\phi\, d\mathbf{a} \\ &= E_{\mathbf{a}_\ell, \phi_\ell} [H(Y_\ell | \mathbf{a}_\ell, \phi_\ell, X_{\ell-1})]. \end{aligned}$$

Also

$$\begin{aligned} H(Y_\ell | X_\ell, C_\ell) &= \int f(\mathbf{a}, \phi) H(Y_\ell | X_\ell, (\mathbf{a}, \phi, X_{\ell-1}))\, d\phi\, d\mathbf{a} \\ &= E_{\mathbf{a}_\ell, \phi_\ell} [H(Y_\ell | \mathbf{a}_\ell, \phi_\ell, X_{\ell-1}^\ell)]. \end{aligned}$$

Therefore

$$\begin{aligned} I(\mathcal{Y}; \mathcal{X} | \mathcal{C}) &= E_{\mathbf{a}_\ell, \phi_\ell} [H(Y_\ell | \mathbf{a}_\ell, \phi_\ell, X_{\ell-1}) - H(Y_\ell | \mathbf{a}_\ell, \phi_\ell, X_{\ell-1}^\ell)] \\ &= E_{\mathbf{a}_\ell, \phi_\ell} [I(Y_\ell; X_\ell | \mathbf{a}_\ell, \phi_\ell, X_{\ell-1})], \end{aligned}$$

which, because the fading-process is ergodic, has the Monte Carlo estimate

$$I(\mathcal{Y}; \mathcal{X} | \mathcal{C}) \approx \frac{1}{N} \sum_{n=1}^N I(Y_n; X_n | \mathbf{a}_n, \phi_n, X_{n-1}).$$

B. Information rate of the FSMC

For the channel modelled as an FSMC with the fading-amplitudes partitioned into discrete regions R_k and the phase-difference ϕ discretised into intervals T_v, we obtain $H(Y_\ell | C_\ell) =$

$$\begin{aligned} &\sum_{k=1}^K \sum_{v=1}^M \Pr(U_\ell = k, V_\ell = v) \sum_{x_{\ell-1} \in \mathcal{X}} \Pr(x_{\ell-1}) \times \\ &\quad H(Y_\ell | U_\ell = k, V_\ell = v, X_{\ell-1} = x_{\ell-1}) \\ &= E_{k,v} [H(Y_\ell | U_\ell = k, V_\ell = v, X_{\ell-1})], \end{aligned}$$

where the channel observation law is given by (7). Also we obtain

$$H(Y_\ell|X_\ell, C_\ell) \;=\; E_{k,v}[H(Y_\ell|U_\ell = k, V_\ell = v, X_{\ell-1}^{\ell}],$$

so that

$$\begin{aligned} I(\mathcal{Y}; \mathcal{X}|\mathcal{C}) \;&=\; E_{k,v}[I(Y_\ell; X_\ell|U_\ell = k, V_\ell = v, X_{\ell-1})] \\ &= \sum_{k=1}^{K}\sum_{v=1}^{M} \pi_{u,v}^{L}\; I(Y_\ell; X_\ell|U_\ell = k, V_\ell = v, X_{\ell-1}). \end{aligned}$$

V. Results

In what follows we shall choose $P(t)$ to be a rectangular pulse of length T_s, defined by

$$P(t) = \begin{cases} \frac{1}{T_s}, & 0 \le t \le T_s; \\ 0, & \text{otherwise,} \end{cases}$$

whereby it is found that

$$P_{\text{total}}(t) = \begin{cases} 1 - \frac{|t|}{T_s}, & -T_s \le t \le T_s; \\ 0, & \text{otherwise.} \end{cases}$$

For a two-tap channel with physical-tap powers $\sigma_g^2(0) = .5$ and $\sigma_g^2(1) = .1$, normalised tap-delays $\tau_0/T_s = .1$ and $\tau_1/T_s = .2$, and cross-correlation of 0.94 between fading at the effective taps, a sequence of one million simulated fading-amplitude vectors $\mathbf{a}$ and phase-differences ϕ was generated using an AR order of 1. Vector-quantisation was then applied to the sequence to define a given number $K = s_T^2$ of FSMC amplitude-states. (The number of ϕ-intervals was set to $M = 50$.)

This FSMC with vector-quantised states was compared with an FSMC that had the same number of amplitude-states, but which were defined without taking the correlation between taps into account, that is, by discretising the amplitudes a_0 and a_1 independently into s_T states for each tap. The ideal-CSI information rates of the two classes of FSMCs, for various numbers of states, were approximated via Monte Carlo from the above simulated sequence, and are shown in Figure 2.

It is seen that for all numbers of amplitude-states above $K = 1$, and for all SNR (signal-to-noise ratio) values between 0 and 20, the FSMC with independent states at each tap is outperformed by the vector-quantised FSMC. It is also seen that the information rate of the vector-quantised FSMC closely approaches that of the continuous-state channel as K increases.

VI. Conclusions

For the frequency-selective fading channel with correlated fading at different taps, we have demonstrated that vector-quantisation of the correlated space of amplitudes allows FSMC amplitude-states to be defined advantageously. Specifically, when compared with an FSMC for which the states are defined independently for each tap, the FSMC with vector-quantised states possesses a greater measure of fidelity to the fading channel for the same number of states (and conversely requires fewer states for the same level of accuracy).

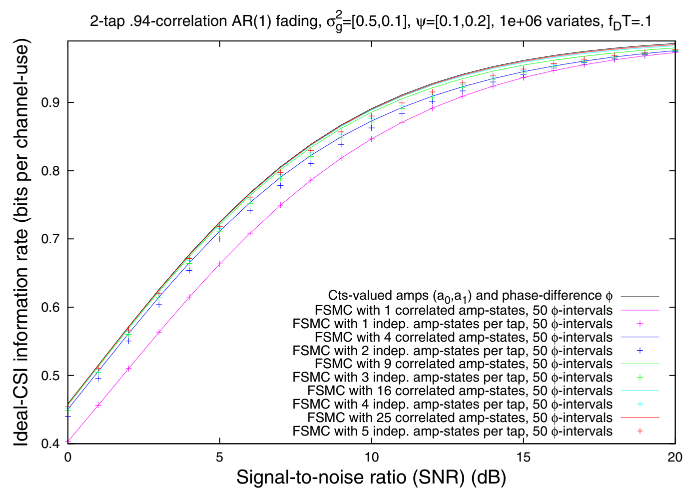

Fig. 2. Comparison of ideal-CSI information rates of the correlated- and uncorrelated-states FSMCs

The application to vector-quantisation of some kind of heuristic guided by knowledge of the correlation between taps, in place of the general k-means clustering algorithm, might lead to improvements in both the information rates and the time required for their computation.

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