

# Static Output Feedback Pole Placement: a Trust Region Approach using the Characteristic Polynomial

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**Abstract**—The recent paper [1] presents a pair of trust region based algorithms for the problem of pole placement via static output feedback. These notes revisit this problem and present an alternate, closely related algorithm. The main change is that instead of having a cost function based on the eigenvalues of the closed loop system matrix, the cost function is based on the characteristic polynomial of the closed loop system. The main advantage of the new approach is that, unlike for the prior algorithms, desired poles no longer have to be distinct.

**Index Terms**—Pole placement, static output feedback, characteristic polynomial, trust region method, Levenberg-Marquardt method.

## I. MAIN IDEA

In [1] the cost function to be minimized is based on the eigenvalues of the closed loop system matrix; the optimization problem is

$$\min_{K \in \mathbb{R}^{m \times p}} f(K) := \frac{1}{2} \|\lambda(A + BKC) - \lambda^D\|_2^2.$$

Here  $A$ ,  $B$  and  $C$  are the system matrices,  $K$  is the output feedback controller matrix we wish to find,  $\lambda^D$  is the vector of desired poles, and  $\lambda(A + BKC)$  denotes the vector of eigenvalues of  $A + BKC$ , with entries sorted to give the minimum norm. As discussed in [1], the eigenvalues of  $A + BKC$  may not be differentiable everywhere; problems can occur at points where  $A + BKC$  has repeated eigenvalues. A consequence of this is that the algorithms in [1] are only appropriate for problems with distinct desired poles (though strategies for dealing with repeated poles were presented).

In these notes we consider a different optimization problem based on the characteristic polynomial. Given  $A$ ,  $B$  and  $C$ , let  $\beta_i = \beta_i(K)$ ,  $i = 1, \dots, n$ , denote the coefficients of the characteristic polynomial of  $A + BKC$ :

$$\det(sI - A - BKC) = s^n + \beta_1 s^{n-1} + \dots + \beta_{n-1} s + \beta_n.$$

Given a desired characteristic polynomial

$$s^n + \beta_1^D s^{n-1} + \dots + \beta_{n-1}^D s + \beta_n^D,$$

we are interested in solving the following unconstrained nonlinear least squares problem

$$\min_{K \in \mathbb{R}^{m \times p}} f(K) := \frac{1}{2} \sum_{i=1}^n |\beta_i(K) - \beta_i^D|^2. \quad (1)$$

Note that the  $\beta_i(K)$ 's depend smoothly on  $K$  and hence that this new problem is smooth.

Such problems can be solved by employing the Levenberg-Marquardt method, a type of trust region method. (An overview of trust region methods can be found in [1].) The only missing ingredients are a means of efficiently evaluating the  $\beta_i(K)$ 's and their first order derivatives. It turns out that these quantities can be calculated via fairly simple expressions [2]. For example,

$$\frac{\partial \beta_i}{\partial K} = - \sum_{j=0}^{i-1} \beta_j [C(A + BKC)^{i-j-1} B]^T$$

with  $\beta_0 = 1$ .

## II. COMPUTATIONAL RESULTS

Unlike the algorithms in [1], the algorithm in these notes has not yet been extensively tested, though preliminary tests indicate that it performs very well for both problems with or without repeated poles. To test how well the algorithm works on problems with repeated poles, we considered the purely academic problem considered in [3]. The system matrices for the problem are as follows,

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & -1 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -7 & 0 & 0 & -2 \\ 0 & -1 & 0 & 1 & 4 & 0 & 0 & 2 \\ 0 & -2 & 0 & 0 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & -2 \\ 0 & -1 & 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ -1 & -1 & -3 \\ 1 & 0 & 1 \\ 0 & 2 & 4 \\ 2 & 1 & 5 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and the aim is to place all poles at  $-1$ .

The algorithm can find a number of solutions to this problem. Each solution usually takes less than 2 seconds to find. One solution is

$$K = \begin{bmatrix} -37.55570287615051 & -55.41114535598239 & -23.65473746507896 \\ -33.63314577143588 & 34.65994081588856 & -23.54425452967808 \\ 20.67875199475054 & -5.11986091092751 & 13.95557129900548 \end{bmatrix}.$$

As is well known, see for example [4, Section 7.2.3], if a matrix has repeated eigenvalues, small perturbations to the

matrix can lead to large changes in its eigenvalues. Solutions to the above problem are quite sensitive; for the above  $K$ ,

$$\left(\sum_{i=1}^n |\beta_i(K) - \beta_i^D|^2\right)^{\frac{1}{2}}$$

is less than  $10^{-10}$  while

$$\left(\sum_{i=1}^n |\lambda_i(A + BKC) + 1|^2\right)^{\frac{1}{2}}$$

is approximately 0.15, rather than being close to 0. (Part of the problem may be difficulty in accurately calculating the eigenvalues of  $A+BKC$ .)

Other problems with repeated desired poles are not so sensitive. For instance, the repeated eigenvalue problem considered in Section V.C of [1] can easily be solved, giving solutions that satisfy  $\|\lambda(A + BKC) - \lambda^D\| < 10^{-6}$ .

### III. ADDITIONAL COMMENTS

Regarding convergence properties of the algorithm, as the cost function is smooth, the general trust region convergence properties in Section II.B of [1] apply; unlike for the algorithms in [1], no qualifications are required here.

The algorithm minimizes distances between characteristic polynomials rather than the real objects of interest, the eigenvalues. If the algorithm converges to a local solution with non-zero cost, even if  $f(K)$  in (1) is small, the distance between the actual and desired eigenvalues may still be fairly large. This is only really an issue if the problem is not solvable or if it is difficult to find a solution.

### REFERENCES

- [1] K. Yang and R. Orsi, "Static output feedback pole placement via a trust region approach," 2007, to appear in *IEEE Trans. Automatic Control*.
- [2] L. F. Godbout and D. Jordan, "Gradient matrices for output feedback systems," *International J. of Control*, vol. 32, no. 3, pp. 411–433, 1980.
- [3] X. A. Wang, "Grassmannian, central projection and output feedback pole assignment of linear systems," *IEEE Trans. Automatic Control*, vol. 41, no. 6, pp. 786–794, 1996.
- [4] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 3rd ed. Baltimore: Johns Hopkins University Press, 1996.