

Tomasz KOWALSKI

A SYNTACTIC PROOF OF A CONJECTURE OF A. WROŃSKI

A syntactic derivation of Cornish identity (J) from the axioms of HBCK is presented which amounts to a syntactic proof of Wroński's conjecture that naturally ordered BCK-algebras form a variety.

The conjecture referred to in the title is the statement that naturally ordered BCK-algebras (HBCK) form a variety. It was proposed by A. Wroński in [1], where he conjectured also that HBCK is the class of sub-reducts of *hoops*. Both the conjectures have recently been proved semantically by W.J. Blok and I.M.A. Ferreirim in [2]. The aim of this paper is to provide a syntactic proof of the first of them. To keep its presentation self-contained, let's recall some definitions and facts.

A structure $\langle A; \oplus, 0, \leq \rangle$, where $\langle A; \oplus, 0 \rangle$ is a commutative monoid and \leq is a partial order on A such that, if $x \leq y$, then $x \oplus z \leq y \oplus z$ and 0 is the least element with respect to \leq , is called a *partially ordered commutative integral monoid*. If moreover $\langle A; \oplus, 0, \leq \rangle$ is *residuated*, i.e. for every $a, b \in A$ there is the smallest element, denoted $a \dot{-} b$ in the set $\{x : a \leq b \oplus x\}$,

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then $\langle A; \dot{-}, \oplus, 0, \leq \rangle$ is called a partially ordered commutative residuated integral monoid, a *pocrim* for short. As the partial order can be recovered from the residuation operation by putting $x \leq y$ iff $x \dot{-} y = 0$, we identify a pocrim with the algebra $\langle A; \dot{-}, \oplus, 0 \rangle$.

Pocrims form a quasivariety defined by the identities:

- (1) $((x \dot{-} y)(x \dot{-} z))(z \dot{-} y) = 0$
- (2) $x \dot{-} 0 = x$
- (3) $0 \dot{-} x = 0$
- (4) $x \dot{-} (y \oplus z) = (x \dot{-} y) \dot{-} z$,

and a quasi-identity

- (5) $x \dot{-} y = 0 = y \dot{-} x \Rightarrow x = y$

The class of $\{\dot{-}, 0\}$ -subreducts of pocrims is known to coincide with the quasivariety of BCK-algebras, i.e. algebras $\langle A; \dot{-}, 0 \rangle$, of the type $\langle 2, 0 \rangle$ satisfying (1)–(3) and (5).

Hereafter we will always omit the $\dot{-}$ symbol, we write for example:

- (i) $xx = 0$
- (ii) $xy \leq x$, i.e. $(xy)x = 0$
- (iii) $x(xy) \leq y$, i.e. $(x(xy))y = 0$
- (iv) $(xy)z = (xz)y$

All the above identities are true in BCK.

A pocrim \mathbf{A} is *naturally ordered* if it satisfies the following condition:

$$x \leq y \text{ iff } \exists z : y = x \oplus z$$

Naturally ordered pocrims, called also *hoops*, form a variety which can be axiomatised by (1)–(3) and

- (6) $x \oplus (yx) = y \oplus (xy)$.

Among the consequences of (6) there is the following identity:

$$(H) \quad (xz)(yz) = (xy)(zy)$$

The subclass of BCK defined by (1)–(3), (5), (H) is called HBCK.

The two authors of [2] have shown, by means of a description of subdirectly irreducible hoops, that HBCK is precisely the class of $\{\dot{-}, 0\}$ -subreducts of hoops, and indeed a variety, as it must satisfy the so-called Cornish identity (cf. e.g. Cornish [3]):

$$(J) \quad x(x(y(yx))) = y(y(x(xy)))$$

from which, in turn, the quasi-identity (4) follows easily.

At the end of their paper the authors state, however, that they have not been able to find a syntactic derivation of (J) from HBCK. What follows is such a derivation.

Let us start with some notational conventions. We will write \bar{x} for $x(xy)$, X for $x(x(y(yx)))$, and \bar{y} for $y(yx)$, Y for $y(y(x(xy)))$.

Observe some easy facts:

- (a) $\bar{x} \leq x; \bar{x} \leq y$
- (b) $X \leq \bar{x}; X \leq \bar{y}$
- (c) $(xy)(y\bar{x}) = xy$
- (d) $(x\bar{y})(yx) = xy$
- (e) $\bar{x}\bar{y} \leq yx; XY \leq \bar{y}\bar{x} \leq xy$
- (f) $\bar{x}(\bar{x}\bar{y}) = x(x\bar{y}) = X$
- (g) $x(xY) = \bar{x}(\bar{x}Y) = X(XY)$

Proofs.

The first four are straightforward: (a), (b) are true in every BCK-algebra and (c), (d) are simple consequences of (H) and the fact that $xy = x(x(xy)) = x\bar{x}$.

Ad (e). From (d) we have $(x\bar{y})(xy) \leq yx$ which is $\bar{x}\bar{y} \leq yx$; for the second part, assume (f) is proved and substitute \bar{x} for x , \bar{y} for y in (d) to get the desired inequality.

Ad (f). By (H), $(x(xy))((x\bar{y})(xy)) = (x(x\bar{y}))((xy)(x\bar{y}))$, and this equals $(x(x\bar{y}))0$, as $xy \leq x\bar{y}$; so $(x(xy))((x(xy))\bar{y}) = \bar{x}(\bar{x}\bar{y}) = x(x\bar{y}) = X$. Thus, (f) and the second part of (e) are proved.

Ad (g). By (H), $(x(xy))((xY)(xy)) = (x(xY))((xy)(xY)) = (x(xY))0$, and, by (H) again, $(\bar{x}(\bar{x}\bar{y}))((\bar{x}Y)(\bar{x}\bar{y})) = (\bar{x}(\bar{x}Y))((\bar{x}\bar{y})(\bar{x}Y)) = (\bar{x}(\bar{x}Y))0$; so $(x(xy))((x(xy))Y) = \bar{x}(\bar{x}Y) = x(xY)$, and $(\bar{x}(\bar{x}\bar{y}))((\bar{x}(\bar{x}\bar{y}))Y) = \bar{x}(\bar{x}Y) = x(xY)$, which by (f) gives $X(XY) = \bar{x}(\bar{x}Y) = x(xY)$. ■

Now, as $xy =$ by (d) $= (x\bar{y})(yx) \geq (x\bar{y})(y\bar{x}) \geq (xy)(y\bar{x}) =$ by (c) $= xy$, we have:

$$(7) \quad xy = (x\bar{y})(y\bar{x}) = (xX)(yY) = (xX)(y\bar{x}) = (x\bar{y})(yY) \\ yx = (y\bar{x})(x\bar{y}) = (yY)(xX) = (yY)(x\bar{y}) = (y\bar{x})(xX).$$

An easy application of (H), together with the symmetry of the identities involved, leads to the following:

$$(8) \quad xy = (xX)(yX) = (xY)(yY), \quad yx = (yX)(xX) = (yY)(xY).$$

$$(8') \quad xy = (x\bar{y})(yX) = (xY)(y\bar{x}), \quad yx = (yX)(x\bar{y}) = (y\bar{x})(xY).$$

Then, we will proceed to state a useful inequality. Let us calculate the following: $(YX)(xy) = ((y(y\bar{x}))X)(xy) =$ by (iv) and (7) $= ((yX)(y\bar{x}))((xX)(y\bar{x})) =$ by (H) $= ((yX)(xX))((y\bar{x})(xX))$; and this, by (8) and (7) equals $(yx)(yx) = 0$. Thus,

$$(9) \quad YX \leq xy; \quad XY \leq yx.$$

With (8) and (9) some further steps are possible: $((xY)(yX))(xy) = ((xY)(yX))((xX)(yX)) = ((xY)(xX))((yX)(xX)) = (XY)(yx) = 0$, so we have $(xY)(yX) \leq xy$; but this can be strengthened to equality. For, by (8), $xy = (x\bar{y})(yX) \leq (xY)(yX) \leq xy$. Hence:

$$(10) \quad (xY)(yX) = xy.$$

Let us proceed yet further: $(xy)(XY) =$ by (10) $= ((xY)(yX))(XY) = ((x(yX)Y)(XY) =$ by (H) $= ((x(yX))X)(YX) = ((xX)(yX))(YX) = (xy)(YX)$.

And for the next step let's take: $(xY)(yX) = ((xY)(yX))0$, which equals to $((xY)(yX))((y\bar{x})(yX)) =$ by (H) $= ((xY)(y\bar{x}))((yX)(y\bar{x})) =$ by (8') and (iv) $= (xy)(YX)$. Therefore, combining the two above equalities with (10) we can state:

$$(11) \quad (xY)(yX) = xy = (xy)(YX) = (xy)(XY).$$

So, we have: $((xY)(YX))((xy)(YX)) = ((xY)(xy))((YX)(xy))$, and changing the order of X and Y , we obtain $((xY)(XY))((xy)(XY)) = ((xY)(xy))((XY)(xy))$; which by (11), (9), (e) and (iv) lead to:

$$(12) \quad (\bar{x}Y)(YX) = \bar{x}Y; \quad (\bar{x}Y)(XY) = \bar{x}Y.$$

Now, by yet another substitution into (H) we get the following equality: $((xX)(YX))((xy)(YX)) = ((xX)(xy))((YX)(xy))$, the left-hand-side of which equals by (11) and (iv) to $(\bar{x}X)(YX)$ and the right-hand side by (iv) and (9) to $\bar{x}X$. This equality, and the next, obtained by changing the order of X and Y the same way as it was done previously, are then:

$$(13) \quad (\bar{x}X)(YX) = \bar{x}X; \quad (\bar{x}X)(XY) = \bar{x}X.$$

Now, substituting the left-hand-side of the second part of (12) into (H) we obtain: $(\bar{x}Y)(XY) = (\bar{x}X)(YX)$, which by (12) and (13) yields:

$$(14) \quad \bar{x}Y = \bar{x}X.$$

So, we have $\bar{x}(\bar{x}Y) = \bar{x}(\bar{x}X) =$ by (f) $= \bar{x}(\bar{x}(\bar{x}(\bar{x}y))) = X$. But, by (g) $\bar{x}(\bar{x}Y) = X(XY)$, therefore: $X(XY) = X$, which gives: $X(X(XY)) = 0$, i.e. $XY = 0$. This, by symmetry, leads to the desired

$$X = Y.$$

References

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JAGELLONIAN UNIVERSITY
DEPARTMENT OF LOGIC
GRODZKA 52
31-044 KRAKÓW
POLAND